REGULARITY OF HIGHER CODIMENSION AREA MINIMIZING INTEGRAL CURRENTS

EMANUELE SPADARO

Abstract. This lecture notes are an expanded and revised version of the course *Regularity of higher codimension area minimizing integral currents* that I taught at the *ERC-School on Geometric Measure Theory and Real Analysis*, held in Pisa, September 30th - October 30th 2013.

The lectures aim to explain the main steps of a new proof of the partial regularity of area minimizing integer rectifiable currents in higher codimension, due originally to F. Almgren, which is contained in a series of papers in collaboration with C. De Lellis (University of Zürich).

Contents

1. Introduction

The subject of this course is the study of the regularity of *minimal surfaces*, considered in the sense of *area minimizing integer rectifiable currents*. This is a very classical topic and stems from many diverse questions and applications. Among the most known there is perhaps the so called *Plateau problem*, consisting in finding the submanifolds of least possible volume among all those submanifolds with a fixed boundary.

Plateau problem. Let M be a $(m+n)$-dimensional Riemannian manifold and $\Gamma \subset M$ a compact $(m-1)$-dimensional oriented submanifold. Find an m-dimensional oriented submanifold Σ with boundary Γ such that

$$\mathrm{vol}_m(\Sigma) \leq \mathrm{vol}_m(\Sigma'),$$

for all oriented submanifolds $\Sigma' \subset M$ such that $\partial \Sigma' = \Gamma$.

It is a well-known fact that the solution of the Plateau problem does not always exist. For example, consider $M = \mathbb{R}^4$, $n = m = 2$ and Γ the smooth Jordan curve parametrized in the following way:

$$\Gamma = \big\{(\zeta^2, \zeta^3) : \zeta \in \mathbb{C},\ |\zeta| = 1\big\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

where we use the usual identification between $\mathbb{C}^2$ and $\mathbb{R}^4$, and we choose the orientation of Γ induced by the anti-clockwise orientation of the unit circle $|\zeta| = 1$ in $\mathbb{C}$. It can be shown (and we will come back to this point in the next sections) that there exist no smooth solutions to the Plateau problem for such fixed boundary, and the *(singular) immersed* 2-dimensional disk

$$S = \big\{(z, w)\ :\ z^3 = w^2,\ |z| \leq 1\big\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

oriented in such a way that $\partial S = \Gamma$, satisfies

$$\mathcal{H}^2(S) < \mathcal{H}^2(\Sigma),$$

for all smooth, oriented 2-dimensional submanifolds $\Sigma \subset \mathbb{R}^4$ with $\partial \Sigma = \Gamma$. Here and in the following we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, which for $k \in \mathbb{N}$ corresponds to the ordinary k-volume on smooth k-dimensional submanifolds.

This fact motivates the introduction of *weak solutions* to the Plateau problem, and the main questions about their existence and regularity.

1.1. **Integer rectifiable currents.** One of the most successful theories of generalized submanifolds is the one by H. Federer and W. Fleming in [19] on integer rectifiable currents (see also [8, 9] for the special case of codimension one generalized submanifolds). From now on, in order to keep the technicalities to a minimum level, we assume that our ambient Riemannian manifold M is Euclidean.

Definition 1.1.1 (Integer rectifiable currents)**.** An integer rectifiable current T of dimension m in $\mathbb{R}^{m+n}$ is a triple $T = (R, \tau, \theta)$ such that:

(i) R is a *rectifiable set*, i.e. $R = \bigcup_{i\in\mathbb{N}} C_i$ with $\mathcal{H}^m(R_0) = 0$ and $C_i \subset M_i$ for every $i \in \mathbb{N} \setminus \{0\}$, where M_i are m-dimensional oriented $C^1 1$ submanifolds of $\mathbb{R}^{m+n}$;

(ii) $\tau : R \to \Lambda_m$ is a measurable map, called *orientation*, taking values in the space of m-vectors such that, for $\mathcal{H}^m$-a.e. $x \in C_i$, $\tau(x) = v_1 \wedge \cdots \wedge v_m$ with $\{v_1, \ldots, v_m\}$ an oriented orthonormal basis of $T_x M_i$;

(iii) $\theta : R \to \mathbb{Z}$ is a measurable function, called *multiplicity*, which is integrable with respect to $\mathcal{H}^m$.

An integer rectifiable current $T = (R, \tau, \theta)$ induces a continuous linear functional (with respect to the natural Fréchet topology) on smooth, compactly supported m-dimensional differential forms ω, denoted by $\mathscr{D}^m$, acting as follows

$$T(\omega) = \int_R \theta \, \langle \omega, \tau \rangle \, d\mathcal{H}^m.$$

Remark 1.1.2. The continuous linear functionals defined in the Fréchet space $\mathscr{D}^m$ are called *m-dimensional currents*.

Remark 1.1.3. Note that the submanifold M_i in Definition 1.1.1 are only C^1 regular. This restriction is not redundant, but it is connected to several aspects of the theory of rectifiable sets.

For an integer rectifiable current T, one can define the analog of the boundary and the volume for smooth submanifolds.

Definition 1.1.4 (Boundary and mass)**.** Let $T = (R, \tau, \theta)$ be an integer rectifiable current in $\mathbb{R}^{m+n}$ of dimension m. The *boundary* of T is defined as the $(m-1)$-dimensional current acting as follows

$$\partial T(\omega) := T(d\omega) \quad \forall\, \omega \in \mathscr{D}^{m-1}.$$

The *mass* of T is defined as the quantity

$$\mathbf{M}(T) := \int_R |\theta| \, d\mathcal{H}^m.$$

Note that, in the case $T = (\Sigma, \tau_\Sigma, 1)$ is the current induced by an oriented submanifold Σ with boundary $\partial\Sigma$, with τ_Σ a continuous orienting vector for Σ and similarly $\tau_{\partial\Sigma}$ for its boundary, then by Stoke's Theorem $\partial T = (\partial\Sigma, \tau_{\partial\Sigma}, 1)$ and $\mathbf{M}(T) = \mathrm{vol}_m(\Sigma)$.

Finally we recall that the space of currents is usually endowed with the weak* topology (often called in this context *weak* topology).

Definition 1.1.5 (Weak topology)**.** We say that a sequence of currents $(T_l)_{l\in\mathbb{N}}$ weakly converges to some current T, and we write $T_l \rightharpoonup T$, if

$$T_l(\omega) \to T(\omega) \quad \forall\, \omega \in \mathscr{D}^m.$$

The Plateau problem has now a straightforward generalization in this context of integer rectifiable currents.

Generalized Plateau problem. Let Γ be a compactly supported $(m-1)$-dimensional integer rectifiable current in $\mathbb{R}^{m+n}$ with $\partial\Gamma = 0$. Find an m-dimensional integer rectifiable current T such that $\partial T = \Gamma$ and

$$\mathbf{M}(T) \leq \mathbf{M}(S),$$

for every S integer rectifiable with $\partial S = \Gamma$.

The success of the theory of integer rectifiable currents is linked ultimately to the possibility to solve the generalized Plateau problem, due to the closure theorem by H. Federer and W. Fleming proven in their pioneering paper [19].

Theorem 1.1.6 (Federer and Fleming [19])**.** *Let $(T_l)_{l\in\mathbb{N}}$ be a sequence of m-dimensional integer rectifiable currents in $\mathbb{R}^{m+n}$ with*

$$\sup_{l\in\mathbb{N}} \big(\mathbf{M}(T_l) + \mathbf{M}(\partial T_l)\big) < +\infty,$$

and assume that $T_l \rightharpoonup T$. Then, T is an integer rectifiable current.

It is then natural to ask about the regularity properties of the solutions to the generalized Plateau problem, called in the sequel *area minimizing* integer rectifiable currents.

1.2. **Partial regularity in higher codimension.** The regularity theory for area minimizing integer rectifiable currents depends very much on the dimension of the current and its *codimension* in the ambient space (i.e., using the same letters as above, if T is an m-dimensional current in $\mathbb{R}^{m+n}$, the codimension is n).

In this course we are interested in the general case of currents with higher codimensions $n > 1$. The case $n = 1$ is usually treated separately, because different techniques can be used and more refined results can be proven (see [10, 20, 30, 32, 33, 28] for the interior regularity and [3, 23] for the boundary regularity). In higher codimension the most general result is due to F. Almgren [5] and concerns the interior partial regularity up to a (relatively) closed set of dimension at most $m-2$.

Theorem 1.2.1 (Almgren [5])**.** *Let T be an m-dimensional area minimizing integer rectifiable current in $\mathbb{R}^{m+n}$. Then, there exists a closed set* $\mathrm{Sing}(T)$ *of Hausdorff dimension at most $m-2$ such that in $\mathbb{R}^{m+n} \setminus (\mathrm{spt}\,(\partial T) \cup \mathrm{Sing}(T))$ the current T is induced by the integration over a smooth oriented submanifold of $\mathbb{R}^{m+n}$.*

In the next pages I will give an overview of the new proof of Theorem 1.2.1 given in collaboration with C. De Lellis in a series of papers [13, 16, 17, 14, 15]. Although our proof is considerably simpler than the original one, it remains quite involved: this text is, therefore, meant as a survey of the techniques and the various steps of the proof, and can be considered an introduction to the reading of the papers [17, 14, 15].

Remark 1.2.2. The interior partial regularity can be proven for integer rectifiable currents in a Riemannian manifold M. In [5] Almgren proves the result for C^5 regular ambient manifolds M, while our papers [17, 14, 15] extend this result to $C^{3,\alpha}$ regular manifolds.

Further notation and terminology. Given an m-dimensional integer rectifiable current $T = (R, \tau, \theta)$, we shall often use the following standard notation:

$$\|T\| := |\theta|\,\mathcal{H}^m \llcorner R, \quad \vec{T} := \tau \quad \text{and} \quad \operatorname{spt}(T) := \operatorname{spt}(\|T\|).$$

The regular and the singular part of a current are defined as follows.

$$\begin{aligned}\operatorname{Reg}(T) := \big\{x \in \operatorname{spt}(T) \;:\; &\operatorname{spt}(T) \cap B_r(x) \text{ is induced by a smooth}\\ &\qquad\qquad \text{submanifold for some } r > 0\big\},\\ \operatorname{Sing}(T) := \operatorname{spt}&(T) \setminus \big(\operatorname{spt}(\partial T) \cup \operatorname{Reg}(T)\big).\end{aligned}$$

Acknowledgements. I am very grateful to A. Marchese, for reading a first draft of these lecture notes and suggesting many precious improvements.

2. The blowup argument: a glimpse of the proof

The main idea of the proof of Theorem 1.2.1 is to detect the singularities of an area minimizing current by a blowup analysis. For any $r > 0$ and $x \in \mathbb{R}^{m+n}$, let $\iota_{x,r}$ denote the map

$$\iota_{x,r} : y \mapsto \frac{y - x}{r},$$

and set $T_{x,r} := (\iota_{x,r})_\sharp T$, where $\sharp$ is the push-forward operator, namely

$$(\iota_{x,r})_\sharp T(\omega) := T(\iota_{x,r}^*\omega) \quad \forall\, \omega \in \mathscr{D}^m.$$

By the classical monotonicity formula (see, e.g., [2, Section 5]), for every $r_k \downarrow 0$ and $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$, there exists a subsequence (not relabeled) such that

$$T_{x,r_k} \rightharpoonup S,$$

where S is a cone without boundary (i.e. $S_{0,r} = S$ for all $r > 0$ and $\partial S = 0$) which is locally area minimizing in $\mathbb{R}^{m+n}$. Such a cone will be called, as usual, *a tangent cone to T at x*.

The idea of the blowup analysis dates back to De Giorgi's pioneering paper [10] and has been used in the context of codimension one currents to

recognize singular points and regular points, because in this case the tangent cones to singular and regular points are in fact different.

2.1. **Flat tangent cones do not imply regularity.** This is not the case for higher codimension currents. In order to illustrate this point, let us consider the current $T_{\mathscr{V}}$ induced by the complex curve considered above:

$$\mathscr{V} = \big\{(z,w)\ :\ z^3 = w^2,\ |z| \leq 1\big\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4.$$

It is simple to show that $T_{\mathscr{V}}$ is an area minimizing integer rectifiable current (cp. [18, 5.4.19]), which is singular in the origin. Nevertheless, the unique tangent cone to $T_{\mathscr{V}}$ at 0 is the current $S = (\mathbb{R}^2 \times \{0\}, e_1 \wedge e_2, 2)$ which is associated to the integration on the horizontal plane $\mathbb{R}^2 \times \{0\} \simeq \{w = 0\}$ with multiplicity two. The tangent cone is actually regular, although the origin is a singular point!

2.2. **Non-homogeneous blowup.** One of the main ideas by Almgren is then to extend this reasoning to different types of blowups, by rescaling differently the "horizontal directions", namely those of a flat tangent cone at the point, and the "vertical" ones, which are the orthogonal complement to the former. In this way, in place of preserving the geometric properties of the rectifiable current T, one is led to preserve the *energy* of the associated *multiple valued function*.

In order to explain this point, let us consider again the current $T_{\mathscr{V}}$. The support of such current, namely the complex curve $\mathscr{V}$, can be viewed as the graph of a function which associates to any $z \in \mathbb{C}$ with $|z| \leq 1$ *two* points in the w-plane:

$$z \mapsto \{w_1(z), w_2(z)\} \quad \text{with } w_i(z)^2 = z^3 \text{ for } i = 1, 2. \tag{2.1}$$

Then the right rescaling according to Almgren is the one producing in the limit a multiple valued *harmonic function* preserving the *Dirichlet energy* (for the definitions see the next sections). In the case of $\mathscr{V}$, the correct rescaling is the one fixing $\mathscr{V}$. For every $\lambda > 0$, we consider $\Phi_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\Phi_\lambda(z,w) = (\lambda^2\, z, \lambda^3\, w),$$

and note that $(\Phi_\lambda)_\sharp T_{\mathscr{V}} = T_{\mathscr{V}}$ for every $\lambda > 0$. Indeed, in the case of $\mathscr{V}$ the functions w_1 and w_2, being the two determinations of the square root of z^3, are already harmonic functions (at least away from the origin).

2.3. **Multiple valued functions.** Following these arguments, we have then to face the problem of defining harmonic multiple valued functions, and to study their singularities. Abstracting from the above example, we consider the multiple valued functions from a domain in $\mathbb{R}^m$ which take a fixed number $Q \in \mathbb{N} \setminus \{0\}$ of values in $\mathbb{R}^n$. This functions will be called in the sequel *Q-valued functions*.

The definition of harmonic Q-valued functions is a simple issue around any "regular point" $x_0 \in \mathbb{R}^m$, for it is enough to consider just the superposition of

classical harmonic functions (possibly with a constant integer multiplicity), i.e.

$$\mathbb{R}^m \supset B_r(x_0) \ni x \mapsto \{u_1(x), \ldots, u_Q(x)\} \in (\mathbb{R}^n)^Q, \tag{2.2}$$

with u_i harmonic and either $u_i = u_j$ or $u_i(x) \neq u_i(x)$ for every $x \in B_r(x_0)$.

The issue becomes much more subtle around the singular points. As it is clear from the example (2.1), in a neighborhood of the origin there is no representation of the map $z \mapsto \{w_1(z), w_2(z)\}$ as in (2.2). In this case the two values $w_1(z)$ and $w_2(z)$ cannot be ordered in a consistent way (due to the *branch point* at 0), and hence cannot be distinguished one from the other. We are then led to consider a multiple valued function as a map taking Q values in the quotient space $(\mathbb{R}^n)^Q/\sim$ induced by the symmetric group $\mathbf{S}_Q$ of permutation of Q indices: namely, given points $P_i, S_i \in \mathbb{R}^n$,

$$(P_1, \ldots, P_Q) \sim (S_1, \ldots, S_Q)$$

if there exists $\sigma \in \mathbf{S}_Q$ such that $P_i = S_{\sigma(i)}$ for every $i = 1, \ldots, Q$.

Note that the space $(\mathbb{R}^n)^Q/\sim$ is a *singular metric space* (for a naturally defined metric, see the next section). Therefore, harmonic maps with values in $(\mathbb{R}^n)^Q/\sim$ have to be carefully defined, for instance by using the metric theory of harmonic functions developed in [22, 25, 26] (cp. also [13, 27]).

Remark 2.3.1. Note that the integer rectifiable current induced by the graph of a Q-valued function (under suitable hypotheses, cp. [16, Proposition 1.4]) belongs to a subclass of currents, sometimes called "positively oriented", i.e. such that the tangent planes make at almost every point a positive angle with a fixed plane. Nevertheless, as it will become clear along the proof, it is enough to consider this subclass as model currents in order to conclude Theorem 1.2.1.

2.4. **The need of centering.** A major geometric and analytic problem has to be addressed in the blowup procedure sketched above. In order to make it apparent, let us discuss another example. Consider the complex curve $\mathscr{W}$ given by

$$\mathscr{W} = \big\{(z, w) \;:\; (w - z^2)^2 = z^5,\ |z| \leq 1\big\} \subset \mathbb{C}^2.$$

As before, $\mathscr{W}$ can be associated to an area minimizing integer rectifiable current $T_{\mathscr{W}}$ in $\mathbb{R}^4$, which is singular at the origin. It is easy to prove that the unique tangent plane to $T_{\mathscr{W}}$ at 0 is the plane $\{w = 0\}$ taken with multiplicity two. On the other hand, by simple analytical considerations, the only nontrivial inhomogeneous blowup in these vertical and horizontal coordinates is given by

$$\Phi_\lambda(z, w) = (\lambda\, z, \lambda^2\, w),$$

and $(\Phi_\lambda)_\sharp T_{\mathscr{W}}$ converges as $\lambda \to +\infty$ to the current induced by the *smooth* complex curve $\{w = z^2\}$ taken with multiplicity two. In other words, the inhomogeneous blowup did not produce in the limit any singular current and cannot be used to study the singularities of $T_{\mathscr{W}}$.

For this reason it is essential to "renormalize" $T_{\mathscr{W}}$ by averaging out its regular first expansion, on top of which the singular branching behavior happens. In the case we handle, the regular part of $T_{\mathscr{W}}$ is exactly the smooth complex curve $\{w = z^2\}$, while the singular branching is due to the determinations of the square root of z^5. It is then clear why one can look for parametrizations of $\mathscr{W}$ defined in $\{w = z^2\}$, so that the singular map to be considered reduces to

$$z \mapsto \{u_1(z), u_2(z)\} \quad \text{with } u_1(z)^2 = z^5.$$

The regular surface $\{w = z^2\}$ is called *center manifold* by Almgren, because it behaves like (and in this case it is exactly) the average of the sheets of the current in a suitable system of coordinates. In general the determination of the center manifold is not straightforward as in the above example, and actually constitutes the most intricate part of the proof.

2.5. **Excluding an infinite order of contact.** Having taken care of the geometric problem of the averaging, in order to be able to perform successfully the inhomogeneous blowup, one has to be sure that the first singular expansion of the current around its regular part does not occur with an infinite order of contact, because in that case the blowup would be by necessity zero.

This issue involves one of the most interesting and original ideas of F. Almgren, namely a new monotonicity formula for the so called *frequency function* (which is a suitable ratio between the energy and a zero degree norm of the function parametrizing the current). This is in fact the right monotone quantity for the inhomogeneous blowups introduced before, and it allows to show that the first singular term in the "expansion" of the current does not occur with infinite order of contact and actually leads to a nontrivial limiting current.

2.6. **The persistence of singularities.** Finally, in order to conclude the proof we need to assure that the singularities of the current do transfer to singularities of the limiting multiple valued function, which can be studied with more elementary techniques. This is in general not true in a pointwise sense, but it becomes true in a measure theoretic sense as soon as the singular set is supposed to have positive $\mathcal{H}^{m-2+\alpha}$ measure, for some $\alpha > 0$.

The contradiction is then reached in the following way: starting from an area minimizing current with a big singular set ($\mathcal{H}^{m-2+\alpha}$ positive measure), one can perform the analysis outlined before and will end up with a multiple valued function having a big set of singularities, thus giving the desired contradiction.

2.7. **Sketch of the proof.** The rigorous proof of Theorem 1.2.1 is actually much more involved and complicated than the rough outline given in the previous section, and can be found either in [5] or in the recent series of papers [13, 16, 17, 14, 15]. In this lecture notes we give some more details of this recent new proof, and comments on some of the subtleties which were hidden in the general discussion above. Since the proof is very lengthly, we start with a description of the strategy.

The proof is done by contradiction. We will, indeed, always assume the following in the sequel.

Contradiction assumption: there exist numbers $m \geq 2$, $n \geq 1$, $\alpha > 0$ and an area minimizing m-dimensional integer rectifiable current T in $\mathbb{R}^{m+n}$ such that

$$\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T)) > 0.$$

Note that the hypothesis $m \geq 2$ is justified because, for $m = 1$ an area minimizing current is locally the union of finitely many non-intersecting open segments.

The aim of the proof is now to show that there exist suitable points of $\mathrm{Sing}(T)$ where we can perform the blowup analysis outlined in the previous section. This process consists of different steps, which we next list in a way which does not require the introduction of new notation but needs to be further specified later.

(A) Find a point $x_0 \in \mathrm{Sing}(T)$ and a sequence of radii $(r_k)_k$ with $r_k \downarrow 0$ such that:

- (A_1) the rescaling currents $T_{x_0,r_k} := (\iota_{x_0,r_k})_\sharp T$ converge to a flat tangent cone;
- (A_2) $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T_{x_0,r_k}) \cap B_1) > \eta > 0$ for some $\eta > 0$ and for every $k \in \mathbb{N}$.

Note that both conclusions hold for suitable subsequences, which in principle may not coincide. What we need to prove is that we can select a point and a subsequence satisfying both.

(B) Construction of the center manifold $\mathcal{M}$ and of a normal Lipschitz approximation $N : \mathcal{M} \to \mathbb{R}^{m+n}/\sim$.

This is the most technical part of the proof, and most of the conclusions of the next steps will intimately depend on this construction.

(C) The center manifold that one constructs in step (B) can only be used in general for a finite number of radii r_k of step (A). The reason is that in general its degree of approximation of the average of the minimizing currents T is under control only up to a certain distance from the singular point under consideration. This leads us to define the sets where the approximation works, called in the sequel *intervals of flattening*, and to define an entire *sequence of center manifolds* which will be used in the blowup analysis.

(D) Next we will take care of the problem of the infinite order of contact. This is done in two part. For the first one we derive the *almost monotonicity formula* for a variant of Almgren's frequency function, deducing that the order of contact remains finite within each center manifold of the sequence in (C).

(E) Then one needs to compare different center manifolds and to show that the order of contact still remains finite. This is done by exploiting a deep consequence of the construction in (C) which we call *splitting before tilting* after the inspiring paper by T. Rivière [29].

(F) With this analysis at hand, we can pass into the limit our blowup sequence and conclude the convergence to the graph of a harmonic Q-valued function u.

(G) Finally, we discuss the capacitary argument leading to the persistence of the singularities, to show that the function u in (F) needs to have a singular set with positive $\mathcal{H}^{m-2+\alpha}$ measure, thus contradicting the partial regularity estimate for such multiple valued harmonic functions.

In the remaining part of this course we give a more detailed description of the steps above, referring to the original papers [13, 16, 17, 14, 15] for the complete proofs.

3. Q-valued functions and rectifiable currents

Since the final contradiction argument relies on the regularity theory of multiple valued functions, we start recalling the main definitions and results concerning them, and the way they can be used to approximate integer rectifiable currents. The reference for this part of the theory is [13, 16, 17, 34].

3.1. **Q-valued functions.** We start by giving a metric structure to the space $(\mathbb{R}^n)^Q/\sim$ of unordered Q-tuples of points in $\mathbb{R}^n$, where $Q \in \mathbb{N} \setminus \{0\}$ is a fixed number. It is immediate to see that this space can be identified with the subset of positive measures of mass Q which are the sum of integer multiplicity Dirac delta:

$$(\mathbb{R}^n)^Q/\sim \quad \simeq \quad \mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \;:\; P_i \in \mathbb{R}^n \right\},$$

where $[\![P_i]\!]$ denotes the Dirac delta at P_i. We can then endow $\mathcal{A}_Q$ with one of the distances defined for (probability) measures, for example the Wasserstein distance of exponent two: for every $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$, we set

$$\mathcal{G}(T_1, T_2) := \min_{\sigma \in \mathbf{S}_Q} \sqrt{\sum_{i=1}^{Q} \left|P_i - S_{\sigma(i)}\right|^2},$$

where we recall that $\mathbf{S}_Q$ denotes the symmetric group of Q elements.

A Q-function simply a map $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^m$ is an open domain. We can then talk about measurable (with respect to the Borel σ-algebra of $\mathcal{A}_Q(\mathbb{R}^n)$), bounded, uniformly-, Hölder- or Lipschitz-continuous Q-valued functions.

More importantly, following the pioneering approach to weakly differentiable functions with values in a metric space by L. Ambrosio [6], we can also define the class of Sobolev Q-valued functions $W^{1,2}$.

Definition 3.1.1 (Sobolev Q-valued functions)**.** Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. A measurable function $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,2}$ if there exist m functions $\varphi_j \in L^2(\Omega)$ for $j = 1, \ldots, m$, such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,2}(\Omega)$ for all $T \in \mathcal{A}_Q$;

(ii) $|\partial_j \mathcal{G}(f, T)| \leq \varphi_j$ almost everywhere in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \ldots, m\}$, where $\partial_j \mathcal{G}(f, T)$ denotes the weak partial derivatives of the functions in (i).

By simple reasonings, one can infer the existence of minimal functions $|\partial_j f|$ fulfilling (ii):

$$|\partial_j f| \leq \varphi_j \text{ a.e. for any other } \varphi_j \text{ satisfying (ii),}$$

We set

$$|Df|^2 := \sum_{j=1}^m |\partial_j f|^2 , \tag{3.1}$$

and define the Dirichlet energy of a Q-valued function as (cp. also [25, 26, 27] for alternative definitions)

$$\mathrm{Dir}(f) := \int_\Omega |Df|^2 .$$

A Q-valued function f is said *Dir-minimizing* if

$$\int_\Omega |Df|^2 \leq \int_\Omega |Dg|^2 \tag{3.2}$$
$$\text{for all } g \in W^{1,2}(\Omega, \mathcal{A}_Q) \text{ with } \mathcal{G}(f, g)|_{\partial\Omega} = 0,$$

where the last inequality is meant in the sense of traces.

The main result in the theory of Q-valued functions is the following.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open domain with Lipschitz boundary, and let $g \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be fixed. Then, the following holds.*

(i) *There exists a Dir-minimizing function f solving the minimization problem* (3.2)*.*

(ii) *Every such function f belongs to $C^{0,\kappa}_{\mathrm{loc}}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ for a dimensional constant $\kappa = \kappa(m, Q) > 0$.*

(iii) *For every such function f, $|Df| \in L^p_{\mathrm{loc}}(\Omega)$ for some dimensional constant $p = p(m, n, Q) > 2$.*

(iv) *There exists a relatively closed set* $\mathrm{Sing}(u) \subset \Omega$ *of Hausdorff dimension at most* $m-2$ *such that the graph of* u *outside* $\mathrm{Sing}(u)$*, i.e. the set*

$$\mathrm{graph}(u|_{\Omega\setminus\Sigma} = \{(x,y)\ :\ x \in \Omega \setminus \Sigma,\ y \in \mathrm{spt}\,(u(x))\}\,,$$

is a smoothly embedded m*-dimensional submanifold of* $\mathbb{R}^{m+n}$*.*

Remark 3.1.3. We refer to [13, 34] for the proofs and more refined results in the case of of two dimensional domains. Moreover, for some results concerning the boundary regularity we refer to [24], and for an improved estimate of the singular set to [21].

We close this section by some considerations on the Q-valued functions. For the reasons explained in the previous section, a Q-valued function has to be considered as an intrinsic map taking values in the non-smooth space of Q-points $\mathcal{A}_Q$, and cannot be reduced to a "superposition" of a number Q of functions. Nevertheless, in many situations it is possible to handle Q-valued functions as a superposition. For example, as shown in [13, Proposition 0.4] every measurable function $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ can be written (not uniquely!) as

$$f(x) = \sum_{i=1}^{Q} [\![f_i(x)]\!] \quad \text{for } \mathcal{H}^m\text{-a.e. } x, \tag{3.3}$$

with $f_1, \ldots, f_Q : \mathbb{R}^m \to \mathbb{R}^n$ measurable functions.

Similarly, for weakly differentiable functions it is possible to define a notion of pointwise approximate differential (cp. [13, Corollary 2,7])

$$Df = \sum_i [\![Df_i]\!] \in \mathcal{A}_Q(\mathbb{R}^{n\times m}),$$

with the property that at almost every x it holds $Df_i(x) = Df_j(x)$ if $f_i(x) = f_j(x)$. Note, however, that the functions f_i do not need to be weakly differentiable in (3.3), for the Q-valued function f has an approximate differential.

3.2. **Graph of Lipschitz Q-valued functions.** There is a canonical way to give the structure of integer rectifiable currents to the graph of a Lipschitz Q-valued function.

To this aim, we consider *proper* Q-valued functions, i.e. measurable functions $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ (where M is any m-dimensional submanifold of $\mathbb{R}^{m+n}$) such that there is a measurable selection $F = \sum_i [\![F_i]\!]$ for which

$$\bigcup_i \overline{(F_i)^{-1}(K)}$$

is compact for every compact $K \subset \mathbb{R}^{m+n}$. It is then obvious that if there exists such a selection, then *every* measurable selection shares the same property.

By a simple induction argument (cp. [16, Lemma 1.1]), there are a countable partition of M in bounded measurable subsets M_i ($i \in \mathbb{N}$) and Lipschitz functions $f_i^j : M_i \to \mathbb{R}^{m+n}$ ($j \in \{1, \ldots, Q\}$) such that

(a) $F|_{M_i} = \sum_{j=1}^Q \left[\!\left[f_i^j \right]\!\right]$ for every $i \in \mathbb{N}$ and $\mathrm{Lip}(f_i^j) \leq \mathrm{Lip}(F)$ $\forall i, j$;

(b) $\forall\ i \in \mathbb{N}$ and $j, j' \in \{1, \ldots, Q\}$, either $f_i^j \equiv f_i^{j'}$ or $f_i^j(x) \neq f_i^{j'}(x)$ $\forall x \in M_i$;

(c) $\forall\ i$ we have $DF(x) = \sum_{j=1}^Q \left[\!\left[Df_i^j(x) \right]\!\right]$ for a.e. $x \in M_i$.

We can then give the following definition.

Definition 3.2.1 (Q-valued push-forward)**.** Let M be an oriented submanifold of $\mathbb{R}^{m+n}$ of dimension m and let $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ be a proper Lipschitz map. Then, we define the push-forward of M through F as the current

$$\mathbf{T}_F = \sum_{i,j} (f_i^j)_\sharp \left[\!\left[M_i \right]\!\right],$$

where M_i and f_i^j are as above: that is,

$$\mathbf{T}_F(\omega) := \sum_{i \in \mathbb{N}} \sum_{j=1}^{Q} \int_{M_i} \langle \omega(f_i^j(x)), Df_i^j(x)_\sharp \vec{e}(x) \rangle \, d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathscr{D}^m(\mathbb{R}^n)\,. \tag{3.4}$$

One can prove that the current in Definition 3.2.1 does not depend on the decomposition chosen for M and f, and moreover is integer rectifiable (cp. [16, Proposition 1.4])

A particular class of push-forwards are given by graphs.

Definition 3.2.2 (Q-graphs)**.** Let $f = \sum_i \left[\!\left[f_i \right]\!\right] : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz and define the map $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ as $F(x) := \sum_{i=1}^Q \left[\!\left[(x, f_i(x)) \right]\!\right]$. Then, $\mathbf{T}_F$ is the *current associated to the graph* $\mathrm{Gr}(f)$ and will be denoted by $\mathbf{G}_f$.

The main result concerning the push-forward of a Q-valued function is the following (see [16, Theorem 2.1]).

Theorem 3.2.3 (Boundary of the push-forward)**.** *Let $M \subset \mathbb{R}^{m+n}$ be an m-dimensional submanifold with boundary, $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ a proper Lipschitz function and $f = F|_{\partial M}$. Then, $\partial \mathbf{T}_F = \mathbf{T}_f$.*

Moreover, the following Taylor expansion of the mass of a graph holds (cp. [16, Corollary 3.3]).

Proposition 3.2.4 (Expansion of $\mathbf{M}(\mathbf{G}_f)$)**.** *There exist dimensional constants $\bar{c}, C > 0$ such that, if $\Omega \subset \mathbb{R}^m$ is a bounded open set and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Lipschitz map with $Lip(f) \leq \bar{c}$, then*

$$\mathbf{M}(\mathbf{G}_f) = Q|\Omega| + \frac{1}{2} \int_\Omega |Df|^2 + \int_\Omega \sum_i \bar{R}_4(Df_i)\,, \tag{3.5}$$

where $\bar{R}_4 \in C^1(\mathbb{R}^{n\times m})$ *satisfies* $|\bar{R}_4(D)| = |D|^3 \bar{L}(D)$ *for* $\bar{L} : \mathbb{R}^{n\times m} \to \mathbb{R}$ *Lipschitz with* $Lip(\bar{L}) \leq C$ *and* $\bar{L}(0) = 0$.

3.3. **Approximation of area minimizing currents.** Finally we recall some results on the approximation of area minimizing currents.

To this aim we need to introduce more notation. We consider cylinders in $\mathbb{R}^{m+n}$ of the form $\bar{C}_s(x) := \bar{B}_s(x) \times \mathbb{R}^n$ with $x \in \mathbb{R}^m$.

Since we are interested in interior regularity, we can assume for the purposes of this section that we are always in the following setting: for some open cylinder $\bar{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q, the area minimizing current T has compact support in $\bar{C}_{4r}(x)$ and satisfies

$$\mathbf{p}_\sharp T = Q \left[\!\left[\bar{B}_{4r}(x)\right]\!\right] \quad \text{and} \quad \partial T \llcorner \bar{C}_{4r}(x) = 0, \tag{3.6}$$

where $\mathbf{p} : \mathbb{R}^{m+n} \to \pi_0 := \mathbb{R}^m \times \{0\}$ is the orthogonal projection.

We introduce next the main regularity parameter for area minimizing currents, namely the *Excess*.

Definition 3.3.1 (Excess measure)**.** For a current T as above we define the *cylindrical excess* $\mathbf{E}(T, \bar{C}_r(x))$ as follows:

$$\begin{aligned}\mathbf{E}(T, \bar{C}_r(x)) &:= \frac{\|T\|(\bar{C}_r(x))}{\omega_m r^m} - Q \\ &= \frac{1}{2\,\omega_m r^m} \int_{\|T\|(\bar{C}_r(x))} |\vec{T} - \vec{\pi}_0|^2 \,\mathrm{d}\, \|T\|,\end{aligned}$$

where ω_m is the measure of the m-dimensional unit ball, and $\vec{\pi}_0$ is the m-vector orienting π_0.

The most general approximation result of area minimizing currents is the one due to Almgren, and reproved in [17] with more refined techniques, which asserts that under suitable smallness condition of the excess, an area minimizing current coincides on a big set with a graph of a Lipschitz Q-valued function.

Theorem 3.3.2 (Almgren's strong approximation)**.** *There exist constants* $C, \gamma_1, \varepsilon_1 > 0$ *(depending on* m, n, Q*) with the following property. Assume that* T *is area minimizing in the cylinder* $\bar{C}_{4r}(x)$ *and assume that*

$$E := \mathbf{E}(T, \bar{C}_{4\,r}(x)) < \varepsilon_1.$$

Then, there exist a map $f : B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$ *and a closed set* $K \subset \bar{B}_r(x)$ *such that the following holds:*

$$Lip(f) \leq C E^{\gamma_1}, \tag{3.7}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \text{and} \quad |B_r(x) \setminus K| \leq C\, E^{1+\gamma_1}\, r^m, \tag{3.8}$$

$$\left| \|T\|(\bar{C}_r(x)) - Q\,\omega_m\, r^m - \tfrac{1}{2} \int_{B_r(x)} |Df|^2 \right| \leq C\, E^{1+\gamma_1}\, r^m. \tag{3.9}$$

The most important improvement of the theorem above with respect to the preexisting approximation results is the small power E^{γ_1} in the three estimates (3.7) - (3.9). Indeed, this will play a crucial role in the construction of the center manifold. It is worthy mentioning that, when $Q = 1$ and $n = 1$, this approximation theorem was first proved with different techniques by De Giorgi in [10] (cp. also [12, Appendix]).

As a byproduct of this approximation, we also obtain the analog of the so called *harmonic approximation*, which allows us to compare the Lipschitz approximation above with a Dir-minimizing function.

Theorem 3.3.3 (Harmonic approximation)**.** *Let γ_1, ε_1 be the constants of Theorem 3.3.2. Then, for every $\bar\eta > 0$, there is a positive constant $\bar\varepsilon_1 < \varepsilon_1$ with the following property. Assume that T is as in Theorem 3.3.2 and*

$$E := \mathbf{E}(T, \bar C_{4\,r}(x)) < \bar\varepsilon_1.$$

If f is the map in Theorem 3.3.2, then there exists a Dir*-minimizing function w such that*

$$r^{-2}\int_{B_r(x)} \mathcal{G}(f,w)^2 + \int_{B_r(x)} (|Df| - |Dw|)^2 \\ + \int_{B_r(x)} |D(\boldsymbol{\eta}\circ f) - D(\boldsymbol{\eta}\circ w)|^2 \leq \bar\eta\, E\, r^m, \quad (3.10)$$

where $\boldsymbol{\eta} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ is the average map

$$\boldsymbol{\eta}\left(\sum_i [\![P_i]\!]\right) = \frac{1}{Q}\sum_i P_i.$$

4. Selection of contradiction's sequence

In this section we give the details of the first step (A) in § 2.7, namely the selection of a common subsequence such that the rescaled currents converge to a flat tangent cone and the measure of the singular set remains uniformly bounded below away from zero. For this purpose, we introduce the following notation. We denote by $\mathbf{B}_r(x)$ the open ball of radius $r > 0$ in $\mathbb{R}^{m+n}$ (we do not write the point x if the origin) and, for $Q \in \mathbb{N}$, we denote by $\mathrm{D}_Q(T)$ the points of density Q of the current T, and set

$$\mathrm{Reg}_Q(T) := \mathrm{Reg}(T) \cap \mathrm{D}_Q(T) \quad \text{and} \quad \mathrm{Sing}_Q(T) := \mathrm{Sing}(T) \cap \mathrm{D}_Q(T).$$

The precise properties of the sequence that will be used in the blowup argument are stated in the following proposition. We recall that the main hypothesis at the base of the proof is the contradiction assumption of § 2.7, which we restate for reader's convenience.

Contradiction assumption: there exist numbers $m \geq 2$, $n \in \mathbb{N}$, $\alpha > 0$ and an area minimizing m-dimensional integer rectifiable current T in $\mathbb{R}^{m+n}$ such that

$$\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T)) > 0.$$

We introduce the *spherical excess* defined as follows: for a given m-dimensional plane π,

$$\mathbf{E}(T, \mathbf{B}_r(x), \pi) := \frac{1}{2\,\omega_m\, r^m} \int_{\mathbf{B}_r(x)} |\vec{T} - \vec{\pi}|^2 \, d\|T\|,$$
$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau} \mathbf{E}(T, \mathbf{B}_r(x), \tau).$$

Proposition 4.0.4 (Contradiction's sequence). *Under the contradiction assumption, there exist*

(1) *constants $m, n, Q \geq 2$ natural numbers and $\alpha, \eta > 0$ real numbers;*
(2) *an m-dimensional area minimizing integer rectifiable current T in $\mathbb{R}^{m+n}$ with $\partial T = 0$;*
(3) *a sequence $r_k \downarrow 0$*

such that $0 \in \mathrm{D}_Q(T)$ and the following holds:

$$\lim_{k\to+\infty} \mathbf{E}(T_{0,r_k}, \mathbf{B}_{10}) = 0, \tag{4.1}$$
$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{0,r_k}) \cap \mathbf{B}_1) > \eta, \tag{4.2}$$
$$\mathcal{H}^m\big((\mathbf{B}_1 \cap \mathrm{spt}\,(T_{0,r_k})) \setminus \mathrm{D}_Q(T_{0,r_k})\big) > 0 \quad \forall\, k \in \mathbb{N}. \tag{4.3}$$

Here $\mathcal{H}^{m-2+\alpha}_\infty$ is the Hausdorff premeasure computed without any restriction on the diameter of the sets in the coverings.

By Almgren's stratification theorem and by general measure theoretic arguments, there exist sequences satisfying either (4.1) or (4.2). The two subsequences might, however, be different: we show the existence of one point and a single subsequence along which *both* conclusions hold. The proof of the proposition is based on the following two results.

Theorem 4.0.5 (Almgren [5, 2.27]). *Let $\alpha > 0$ and let T be an integer rectifiable area minimizing current in $\mathbb{R}^{m+n}$. Then,*

(1) *for $\mathcal{H}^{m-2+\alpha}$-a.e. point $x \in \mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$ there exists a subsequence $s_k \downarrow 0$ such that T_{x,s_k} converges to a flat cone;*
(2) *for $\mathcal{H}^{m-3+\alpha}$-a.e. point $x \in \mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$, it holds that $\Theta(T, x) \in \mathbb{Z}$.*

Lemma 4.0.6. *Let S be an m-dimensional area minimizing integral current, which is a cone in $\mathbb{R}^{m+n}$ with $\partial S = 0$, $Q = \Theta(S, 0) \in \mathbb{N} \setminus \{0\}$, and assume that*

$$\mathcal{H}^m\big(D_Q(S)\big) > 0 \quad \text{and} \quad \mathcal{H}^{m-1}(\mathrm{Sing}_Q(S)) = 0.$$

Then S is an m-dimensional plane with multiplicity Q.

Proof of Proposition 4.0.4. Let $m > 1$ be the smallest integer for which Theorem 1.2.1 fails. In view of Almgren's stratification Theorem 4.0.5, we can assume that there exist an integer rectifiable area minimizing current R of dimension m and a positive integer Q such that the Hausdorff dimension of $\mathrm{Sing}_Q(R)$ is larger than $m - 2$. We fix the smallest Q for which such a

current R exists and note that by Allard's regularity theorem (cp. [2]) it must be $Q > 1$.

Let $\alpha > 0$ be such that $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(R)) > 0$, and consider a density point x_0 for the measure $\mathcal{H}^{m-2+\alpha}$ (without loss of generality $x_0 = 0$). In particular, there exists $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(R) \cap \mathbf{B}_{r_k}\big)}{r_k^{m-2+\alpha}} > 0\,.$$

Up to a subsequence (not relabeled) we can assume that $R_{0,r_k} \to S$, with S a tangent cone. If S is a multiplicity Q flat plane, then we set $T := R$ and the proposition is proven (indeed, (4.3) is satisfied because $0 \in \mathrm{Sing}(R)$ and $\|R\| \geq \mathcal{H}^m \llcorner \mathrm{spt}\,(R)$).

If S is *not* flat, taking into account the convergence properties of area minimizing currents [31, Theorem 34.5] and the upper semicontinuity of $\mathcal{H}^{m-2+\alpha}_\infty$ under the Hausdorff convergence of compact sets, we deduce

$$\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(S) \cap \bar{\mathbf{B}}_1\big) \geq \liminf_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(R_{0,r_k}) \cap \bar{\mathbf{B}}_1\big) > 0. \tag{4.4}$$

We claim that (4.4) implies

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{Sing}_Q(S)) > 0. \tag{4.5}$$

Indeed, if all points of $\mathrm{D}_Q(S)$ are singular, then (4.5) follows from (4.4) directly. Otherwise, $\mathrm{Reg}_Q(S)$ is not empty, thus implying $\mathcal{H}^m(D_Q(S) \cap \mathbf{B}_1) > 0$: we can then apply Lemma 4.0.6 and infer that, since S is not regular, then $\mathcal{H}^{m-1}(\mathrm{Sing}_Q(S)) > 0$ and (4.5) holds.

We can, hence, find $x \in \mathrm{Sing}_Q(S) \setminus \{0\}$ and $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(S) \cap \mathbf{B}_{r_k}(x)\big)}{r_k^{m-2+\alpha}} > 0.$$

Up to a subsequence (not relabelled), we can assume that S_{x,r_k} converges to S_1. Since S_1 is a tangent cone to the cone S at $x \neq 0$, S_1 splits off a line, i.e. $S_1 = S_2 \times [\![\{t\,e : t \in \mathbb{R}\}]\!]$ for some $e \in \mathbb{S}^{m+n-1}$, for some area minimizing cone S_2 in $\mathbb{R}^{m-1+n}$ and some $v \in \mathbb{R}^{m+n}$ (cp. [31, Lemma 35.5]). Since m is, by assumption, the smallest integer for which Theorem 1.2.1 fails, $\mathcal{H}^{m-3+\alpha}(\mathrm{Sing}(S_2)) = 0$ and, hence, $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(S_1)) = 0$. On the other hand, arguing as for (4.4), we have

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_1) \cap \bar{\mathbf{B}}_1) \geq \limsup_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_{x,r_k}) \cap \bar{\mathbf{B}}_1) > 0.$$

Thus $\mathrm{Reg}_Q(S_1) \neq \emptyset$ and, hence, $\mathcal{H}^m(D_Q(S_1)) > 0$. We then can apply Lemma 4.0.6 again and conclude that S_1 is an m-dimensional plane with multiplicity Q. Therefore, the proposition follows taking T a suitable translation of S. $\square$

Proof of Lemma 4.0.6. We premise the following lemma.

Lemma 4.0.7. *Let T be an integer rectifiable current of dimension m in $\mathbb{R}^{m+n}$ with locally finite mass and U an open set such that*

$$\mathcal{H}^{m-1}(\partial U \cap \operatorname{spt}(T)) = 0 \quad \textit{and} \quad (\partial T)\llcorner U = 0.$$

Then $\partial(T\llcorner U) = 0$.

Proof. Consider $V \subset\subset \mathbb{R}^{m+n}$. By the slicing theory

$$S_r := T\llcorner(V \cap U \cap \{\operatorname{dist}(x, \partial U) > r\})$$

is a normal current in $\mathbf{N}_m(V)$ for a.e. r. Since

$$\mathbf{M}(T\llcorner(V \cap U) - S_r) \to 0 \quad \text{as} \quad r \downarrow 0,$$

we conclude that $T\llcorner(U \cap V)$ is in the $\mathbf{M}$-closure of $\mathbf{N}_m(V)$. Thus, by [18, 4.1.17], $T\llcorner U$ is a flat chain in $\mathbb{R}^{m+n}$ and by [18, 4.1.12] $\partial(T\llcorner U)$ is a flat chain. Since $\operatorname{spt}(\partial(T\llcorner U)) \subset \partial U \cap \operatorname{spt}(T)$, we can apply [18, Theorem 4.1.20] to conclude that $\partial(T\llcorner U) = 0$. □

We next prove Lemma 4.0.6. For each $x \in \operatorname{Reg}_Q(S)$, let r_x be such that $S\llcorner\mathbf{B}_{2r_x}(x) = Q\,[\![\Gamma]\!]$ for some regular submanifold Γ and set

$$U := \bigcup_{x \in \operatorname{Reg}_Q(S)} \mathbf{B}_{r_x}(x).$$

Obviously, $\operatorname{Reg}_Q(S) \subset U$; hence, by assumption, it is not empty. Fix $x \in \operatorname{spt}(S) \cap \partial U$. Let next $(x_k)_{k\in\mathbb{N}} \subset \operatorname{Reg}_Q(S)$ be such that

$$\operatorname{dist}(x, \mathbf{B}_{r_{x_k}}(x_k)) \to 0.$$

We necessarily have that $r_{x_k} \to 0$: otherwise we would have $x \in \mathbf{B}_{2r_{x_k}}(x_k)$ for some k, which would imply $x \in \operatorname{Reg}_Q(S) \subset U$, i.e. a contradiction. Therefore, $x_k \to x$ and, by [31, Theorem 35.1],

$$Q = \limsup_{k\to+\infty} \Theta(S, x_k) \le \Theta(S, x) = \lim_{\lambda\downarrow 0} \Theta(S, \lambda x) \le \Theta(S, 0) = Q.$$

This implies $x \in \mathrm{D}_Q(S)$. Since $x \in \partial U$, we must then have $x \in \operatorname{Sing}_Q(S)$. Thus, we conclude that $\mathcal{H}^{m-1}(\operatorname{spt}(S)\cap\partial U) = 0$. It follows from Lemma 4.0.7 that $S' := S\llcorner U$ has 0 boundary in $\mathbb{R}^{m+n}$. Moreover, since S is an area minimizing cone, S' is also an area-minimizing cone. By definition of U we have $\Theta(S', x) = Q$ for $\|S'\|$-a.e. x and, by semicontinuity,

$$Q \le \Theta(S', 0) \le \Theta(S, 0) = Q.$$

We apply Allard's theorem [2] and deduce that S' is regular, i.e. S' is an m-plane with multiplicity Q. Finally, from $\Theta(S', 0) = \Theta(S, 0)$, we infer $S' = S$. □

5. Center manifold's construction

In this section we describe the procedure for the construction of the center manifold. As mentioned in the introduction, this is the most complicated part of the proof: indeed, the construction of the center manifold comes together with a series of other estimates which will enter significantly in the proof of the main Theorem 1.2.1. In particular, as an outcome of the procedure we obtain the following several things.

(1) A decomposition of the horizontal plane $\pi_0 = \mathbb{R}^m \times \{0\}$ of "Whitney's type".
(2) A family of interpolating functions defined on the cubes of this decomposition.
(3) A normal approximation taking values in the normal bundle of the center manifold.
(4) A set of criteria (which will in fact determine the Whitney decomposition) which lead to what we call *splitting-before-tilting* estimates.
(5) An family of intervals, called *intervals of flattening*, where the construction will be effective.
(6) A family of pairs cube–ball transforming the estimates on the Whitney decomposition into estimates on balls (thus passing from the cubic lattice of the decomposition to the standard geometry of balls).

5.1. **Notation and assumptions.** Let us recall the following notation. Given an integer rectifiable current T with compact support, we consider the *spherical* and the *cylindrical* excesses defined as follows, respectively: for given m-planes π, π', we set

$$\mathbf{E}(T, \mathbf{B}_r(x), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{B}_r(x)} |\vec{T} - \vec{\pi}|^2 \, d\|T\|, \tag{5.1}$$

$$\mathbf{E}(T, \bar{C}_r(x,\pi), \pi') := (2\omega_m\, r^m)^{-1} \int_{\bar{C}_r(x,\pi)} |\vec{T} - \vec{\pi}'|^2 \, d\|T\|\,, \tag{5.2}$$

where $\bar{C}_r(x,\pi) = \bar{B}_r(x,\pi) \times \pi^\perp$ is the cylinder over the closed ball $\bar{B}_r(x,\pi)$ or radius r and center x in the m-dimensional plane π. And we consider the *height function* in a set A (we denote by $\mathbf{p}_\pi$ the orthogonal projection on a plane π)

$$\mathbf{h}(T, A, \pi) := \sup_{x,y \,\in\, \mathrm{spt}\,(T) \,\cap\, A} |\mathbf{p}_{\pi^\perp}(x) - \mathbf{p}_{\pi^\perp}(y)|\,.$$

We also set

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_\tau \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi), \tag{5.3}$$

and we will use $\mathbf{E}(T, \bar{C}_r(x,\pi))$ in place of $\mathbf{E}(T, \bar{C}_r(x,\pi), \pi)$: note that it coincides with the cylindrical excess as defined in § 3.3 when

$$(\mathbf{p}_\pi)_\sharp T \llcorner \bar{C}_r(x,\pi) = Q \left[\!\left[\bar{B}_r(\mathbf{p}_\pi(x), \pi) \right]\!\right].$$

In this section we will work with an area minimizing integer rectifiable current T^0 with compact support which satisfies the following assumptions: for some constant $\varepsilon_2 \in (0,1)$, which we always suppose to be small enough,

$$\Theta(0,T^0) = Q \quad \text{and} \quad \partial T^0 \llcorner \mathbf{B}_{6\sqrt{m}} = 0, \tag{5.4}$$

$$\|T^0\|(\mathbf{B}_{6\sqrt{m}\rho}) \leq \left(\omega_m Q(6\sqrt{m})^m + \varepsilon_2^2\right)\rho^m \quad \forall \rho \leq 1, \tag{5.5}$$

$$E := \mathbf{E}\left(T^0, \mathbf{B}_{6\sqrt{m}}\right) = \mathbf{E}\left(T^0, \mathbf{B}_{6\sqrt{m}}, \pi_0\right) \leq \varepsilon_2^2, \tag{5.6}$$

It follows from standard considerations in geometric measure theory that there are positive constants $C_0(m,n,Q)$ and $c_0(m,n,Q)$ with the following property. If T^0 is as in (5.4) - (5.6), $\varepsilon_2 < c_0$ and $T := T^0 \llcorner \mathbf{B}_{23\sqrt{m}/4}$, then:

$$\partial T \llcorner \bar{C}_{11\sqrt{m}/2}(0,\pi_0) = 0, \tag{5.7}$$

$$(\mathbf{p}_{\pi_0})_\sharp T \llcorner \bar{C}_{11\sqrt{m}/2}(0,\pi_0) = Q \left[\!\!\left[B_{11\sqrt{m}/2}(0,\pi_0) \right]\!\!\right], \tag{5.8}$$

$$\mathbf{h}(T, \bar{C}_{5\sqrt{m}}(0,\pi_0)) \leq C_0 \varepsilon_2^{\frac{1}{m}}\,. \tag{5.9}$$

In particular for each $x \in B_{11\sqrt{m}/2}(0,\pi_0)$ there is a point $p \in \operatorname{spt}(T)$ with $\mathbf{p}_{\pi_0}(p) = x$.

5.2. **Whitney decomposition and interpolating functions.** The construction of the center manifold is done by following a suitable decomposition of the horizontal plane π_0 into cubes. We denote by $\mathscr{C}^j$, $j \in \mathbb{N}$, the family of dyadic closed cubes L of π_0 with side-length $2^{1-j} =: 2\,\ell(L)$. Next we set $\mathscr{C} := \bigcup_{j\in\mathbb{N}} \mathscr{C}^j$. If H and L are two cubes in $\mathscr{C}$ with $H \subset L$, then we call L an *ancestor* of H and H a *descendant* of L. When in addition $\ell(L) = 2\ell(H)$, H is *a son* of L and L *the father* of H.

Definition 5.2.1. A Whitney decomposition of $[-4,4]^m \subset \pi_0$ consists of a closed set $\mathbf{\Gamma} \subset [-4,4]^m$ and a family $\mathscr{W} \subset \mathscr{C}$ satisfying the following properties:

(w1) $\mathbf{\Gamma} \cup \bigcup_{L\in\mathscr{W}} L = [-4,4]^m$ and $\mathbf{\Gamma}$ does not intersect any element of $\mathscr{W}$;
(w2) the interiors of any pair of distinct cubes $L_1, L_2 \in \mathscr{W}$ are disjoint;
(w3) if $L_1, L_2 \in \mathscr{W}$ have nonempty intersection, then

$$\frac{1}{2}\ell(L_1) \leq \ell(L_2) \leq 2\,\ell(L_1).$$

Observe that (w1) - (w3) imply

$$\operatorname{dist}(\mathbf{\Gamma}, L) := \inf\left\{|x-y| : x \in L, y \in \mathbf{\Gamma}\right\} \geq 2\ell(L) \quad \text{for every } L \in \mathscr{W}.$$

However, we do *not* require any inequality of the form $\operatorname{dist}(\mathbf{\Gamma}, L) \leq C\ell(L)$, although this would be customary for what is commonly called Whitney decomposition in the literature.

We denote by $\mathscr{S}^j$ all the dyadic cubes with side-length 2^{1-j} which are not contained in $\mathscr{W}$ and set $\mathscr{S} := \cup_{j\geq N_0} \mathscr{S}^j$ for some big natural number N_0. For each cube $L \in \mathscr{W} \cup \mathscr{S}$, we set $r_L = M_0\sqrt{m}\ell(L)$, with $M_0 \in \mathbb{N}$ a

dimensional constant to be fixed later, and we call its center x_L. We can then find points $p_L \in \mathrm{spt}\,(T)$, with coordinates $p_L = (x_L, y_L) \in \pi_0 \times \pi_0^\perp$, and *interpolating functions*

$$g_L : B_{4r_L}(p_L, \pi_0) \to \pi_0^\perp,$$

such that the following holds: for every $H, L \in \mathscr{W} \cup \mathscr{S}$,

$$\|g_H\|_{C^0} \leq C\,E^{\frac{1}{2m}} \quad \text{and} \quad \|Dg_H\|_{C^{2,\kappa}} \leq CE^{\frac{1}{2}}; \tag{5.10}$$

$$\|g_H - g_L\|_{C^i(B_{r_L}(p_L,\pi_0))} \leq CE^{\frac{1}{2}}\ell(H)^{3+\kappa-i} \tag{5.11}$$

$$\forall\, i \in \{0, \ldots, 3\} \text{ if } H \cap L \neq \emptyset;$$

$$|D^3 g_H(x_H) - D^3 g_L(x_L)| \leq CE^{\frac{1}{2}}|x_H - x_L|^\kappa; \tag{5.12}$$

$$\sup_{(x,y)\in \mathrm{spt}\,(T),\; x\in H} \|g_H - y\|_{C^0} \leq CE^{\frac{1}{2m}}\ell(H), \tag{5.13}$$

for some $\kappa > 0$, and where we used the notation

$$B_r(p_L, \pi_0) := \mathbf{B}_r(p_L) \cap (p_L + \pi_0).$$

It is now very simple to show how to patch all the interpolating functions g_L in order to construct a center manifold. To this aim, we set

$$\mathscr{P}^j := \mathscr{S}^j \cup \left\{L \in \mathscr{W} \;:\; \ell(L) \geq 2^{-j}\right\}.$$

For every $L \in \mathscr{P}^j$ we define

$$\vartheta_L(y) := \vartheta\left(\frac{y - x_L}{\ell(L)}\right),$$

for some fixed $\vartheta \in C_c^\infty\big([-\frac{17}{16}, \frac{17}{16}]^m, [0,1]\big)$ that is identically 1 on $[-1,1]^m$. We can then patch all the interpolating functions using the partition of the unit induced by the ϑ_L, i.e.

$$\varphi_j := \frac{\sum_{L\in\mathscr{P}^j} \vartheta_L g_L}{\sum_{L\in\mathscr{P}^j} \vartheta_L}. \tag{5.14}$$

The following theorem is now a very easy consequence of the estimates on the interpolating functions.

Theorem 5.2.2 (Existence of the center manifold)**.** *Assume to be given a Whitney decomposition $(\Gamma, \mathscr{W})$ and interpolating functions g_H as above. If ε_2 is sufficiently small, then*

(i) *the functions φ_j defined in (5.14) satisfy*

$$\|D\varphi_j\|_{C^{2,\kappa}} \leq CE^{\frac{1}{2}} \quad \textit{and} \quad \|\varphi_j\|_{C^0} \leq CE^{\frac{1}{2m}},$$

(ii) *φ_j converges to a map $\boldsymbol{\varphi}$ such that $\mathcal{M} := \mathrm{Gr}(\boldsymbol{\varphi}|_{]-4,4[^m})$ is a $C^{3,\kappa}$ submanifold of Σ, called in the sequel* center manifold*,*

(iii) *for all $x \in \mathbf{\Gamma}$, the point $(x, \boldsymbol{\varphi}(x)) \in \mathrm{spt}\,(T)$ and is a multiplicity Q point. Setting $\mathbf{\Phi}(y) := (y, \boldsymbol{\varphi}(y))$, we call $\mathbf{\Phi}(\mathbf{\Gamma})$ the* contact set*.*

Proof. Define $\chi_H := \vartheta_H/(\sum_{L\in\mathscr{P}^j}\vartheta_L)$ and observe that

$$\sum \chi_H = 1 \quad \text{and} \quad \|\chi_H\|_{C^i} \le C_0(i,m,n)\,\ell(H)^{-i} \qquad \forall i\in\mathbb{N}\,. \tag{5.15}$$

Set $\mathscr{P}^j(H) := \{L\in\mathscr{P}^j : L\cap H\neq\emptyset\}\setminus\{H\}$. By construction

$$\frac{1}{2}\ell(L) \le \ell(H) \le 2\,\ell(L) \quad \text{for every } L\in\mathscr{P}^j(H),$$

and the cardinality of $\mathscr{P}^j(H)$ is bounded by a geometric constant C_0. The estimate $|\varphi_j| \le C E^{\frac{1}{2m}}$ follows then easily from (5.10).

For $x\in H$ we write

$$\varphi_j(x) = \Big(g_H\chi_H + \sum_{L\in\mathscr{P}^j(H)} g_L\chi_L\Big)(x) = g_H(x) + \sum_{L\in\mathscr{P}^j(H)} (g_L-g_H)\chi_L\,(x)\,. \tag{5.16}$$

Using the Leibniz rule, (5.15), (5.10) and (5.11), for $i\in\{1,2,3\}$ we get

$$\begin{aligned}\|D^i\varphi_j\|_{C^0(H)} &\le \|g_H\|_{C^i} + \sum_{0\le l\le i}\sum_{L\in\mathscr{P}^j(H)} \|g_L-g_H\|_{C^l(H)}\ell(L)^{l-i}\\ &\le C E^{\frac12}\big(1+\ell(H)^{3+\kappa-i}\big).\end{aligned}$$

Next, using also $[D^3 g_H - D^3 g_L]_\kappa \le C E^{\frac12}$, we obtain

$$\begin{aligned}[D^3\varphi_j]_{\kappa,H} \le \sum_{0\le l\le 3}\sum_{L\in\mathscr{P}^j(H)} &\ell(H)^{l-3}\big(\ell(H)^{-\kappa}\|D^l(g_L-g_H)\|_{C^0(H)}\\ &+[D^l(g_L-g_H)]_{\kappa,H}\big) + [D^3 g_H]_{\kappa,H} \le C E^{\frac12}.\end{aligned}$$

Fix now $x,y\in[-4,4]^m$, let $H,L\in\mathscr{P}^j$ be such that $x\in H$ and $y\in L$. If $H\cap L\neq\emptyset$, then

$$|D^3\varphi_j(x) - D^3\varphi_j(y)| \le C\big([D^3\varphi_j]_{\kappa,H} + [D^3\varphi_j]_{\kappa,L}\big)|x-y|^\kappa. \tag{5.17}$$

If $H\cap L=\emptyset$, we assume without loss of generality $\ell(H)\le\ell(L)$ and observe that

$$\max\big\{|x-x_H|,|y-x_L|\big\} \le \ell(L) \le |x-y|\,.$$

Moreover, by construction φ_j is identically equal to g_H in a neighborhood of its center x_H. Thus, we can estimate

$$\begin{aligned}|D^3\varphi_j(x) - D^3\varphi_j(y)| &\le |D^3\varphi_j(x) - D^3\varphi_j(x_H)| + |D^3 g_H(x_H) - D^3 g_L(x_L)|\\ &\quad + |D^3\varphi_j(x_L) - D^3\varphi_j(y)|\\ &\le C E^{\frac12}\left(|x-x_H|^\kappa + |x_H-x_L|^\kappa + |y-x_L|^\kappa\right)\\ &\le C E^{\frac12}|x-y|^\kappa\,,\end{aligned}$$

where we used (5.17) and (5.12). The convergence of the sequence φ_j (up to subsequences) and (iii) are now simple consequences of (5.13) (details are left to the reader). □

5.3. **Normal approximation.** The main feature of the center manifold $\mathcal{M}$ lies actually in the fact that it allows to make a good approximation of the current which turns out to be almost centered by $\mathcal{M}$.

We introduce the following definition.

Definition 5.3.1 ($\mathcal{M}$-normal approximation)**.** An *$\mathcal{M}$-normal approximation* of T is given by a pair $(\mathcal{K}, F)$ such that

(A1) $F : \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is Lipschitz and takes the special form

$$F(x) = \sum_i [\![x + N_i(x)]\!]\,,$$

with $N_i(x) \perp T_x\mathcal{M}$ for every $x \in \mathcal{M}$ and $i = 1, \ldots, Q$.

(A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\mathbf{\Phi}\big(\mathbf{\Gamma} \cap [-\frac{7}{2}, \frac{7}{2}]^m\big)$ and

$$\mathbf{T}_F \llcorner \mathbf{p}^{-1}(\mathcal{K}) = T \llcorner \mathbf{p}^{-1}(\mathcal{K}).$$

The map $N = \sum_i [\![N_i]\!] : \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is called *the normal part* of F.

As proven in [14, Theorem 2.4], the center manifold $\mathcal{M}$ of the previous section allows to construct an $\mathcal{M}$-normal approximation which does approximate the area minimizing current T. In order to state the result, to each $L \in \mathscr{W}$ we associate a *Whitney region* $\mathcal{L}$ on $\mathcal{M}$ as follows:

$$\mathcal{L} := \mathbf{\Phi}\left(H \cap [-\frac{7}{2}, \frac{7}{2}]^m\right),$$

where H is the cube concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$. We will use $\|N|_{\mathcal{L}}\|_0$ to denote the quantity $\sup_{x\in\mathcal{L}} \mathcal{G}(N(x), Q[\![0]\!])$.

Theorem 5.3.2. *Let $\gamma_2 := \frac{\gamma_1}{4}$, with γ_1 the constant of Theorem 3.3.2. Under the hypotheses of Theorem 5.2.2, if ε_2 is sufficiently small, then there exist constants $\beta_2, \delta_2 > 0$ and an $\mathcal{M}$-normal approximation $(\mathcal{K}, F)$ such that the following estimates hold on every Whitney region $\mathcal{L}$:*

$$Lip(N|_{\mathcal{L}}) \leq C E^{\gamma_2} \ell(L)^{\gamma_2} \quad \textit{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leq C E^{\frac{1}{2m}} \ell(L)^{1+\beta_2}, \tag{5.18}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C E\, \ell(L)^{m+2-2\delta_2}, \tag{5.19}$$

$$|\mathcal{L} \setminus \mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C E^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}. \tag{5.20}$$

Moreover, for any $a > 0$ and any Borel set $\mathcal{V} \subset \mathcal{L}$, we have

$$\int_{\mathcal{V}} |\boldsymbol{\eta} \circ N| \leq C E \left(\ell(L)^{3+\frac{\beta_2}{3}} + a\, \ell(L)^{2+\frac{\gamma_2}{2}}\right) |\mathcal{V}| + \frac{C}{a} \int_{\mathcal{V}} \mathcal{G}\big(N, Q[\![\boldsymbol{\eta} \circ N]\!]\big)^{2+\gamma_2}. \tag{5.21}$$

Let us briefly explain the conclusions of the theorem. The estimates in (5.18) and (5.19) concern the regularity properties of the normal approximation N, and will play an important role in many of the subsequent arguments. However, the key properties of N are (5.20) and (5.21): the former

estimates the error done in the approximation on every Whitney region; while the latter estimates the L^1 norm of the average of N, which is a measure of the centering of the center manifold. Note that both estimates are in some sense "superlinear" with respect to the relevant parameters: indeed, as it will be better understood later on, they involve either a superlinear power of the excess $E^{1+\gamma_2}$ or the $L^{2+\gamma_2}$ norm of N (which is of higher order with respect to the "natural" L^2 norm).

5.4. **Construction criteria.** The estimates and the results of the previous two subsections depend very much on the way the Whitney decomposition, the interpolating functions and the normal approximation are constructed.

We start recalling the notation $p_L = (x_L, y_L)$ where L is a dyadic cube, x_L its center and $y_L \in \pi_0^\perp$ is chosen in such a way that $p_L \in \mathrm{spt}\,(T)$. Moreover, we set

$$\mathbf{B}_L := \mathbf{B}_{64 r_L}(p_L),$$

where we recall that $r_L := M_0 \sqrt{m}\, \ell(L)$ for some large constant $M_0 \in \mathbb{N}$.

We define the families of cubes of the Whitney decomposition

$$\mathscr{W} = \mathscr{W}_e \cup \mathscr{W}_h \cup \mathscr{W}_n \quad \text{and} \quad \mathscr{S} \subset \mathscr{C}.$$

We use the notation $\mathscr{S}^j = \mathscr{S} \cap \mathscr{C}^j, \mathscr{W}^j = \mathscr{W} \cap \mathscr{C}^j$ and so on.

We recall the notation for the excess,

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_\tau \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi).$$

The m-dimensional planes π realizing the minimum above are called *optimal planes* of T in a ball $\mathbf{B}_r(x)$ if, in addition, π optimizes the height among all planes that optimize the excess:

$$\begin{aligned} \mathbf{h}(T, \mathbf{B}_r(x), \pi) &= \min \big\{ \mathbf{h}(T, \mathbf{B}_r(x), \tau) : \tau \text{ satisfies } (5.3) \big\} \\ &=: \mathbf{h}(T, \mathbf{B}_r(x)). \end{aligned} \tag{5.22}$$

An optimal plane in the ball $\mathbf{B}_L$ is denoted by π_L.

We fix a big natural number N_0, and constants $C_e, C_h > 0$, and we define $\mathscr{W}^i = \mathscr{S}^i = \emptyset$ for $i < N_0$. We proceed with $j \geq N_0$ inductively: if the father of $L \in \mathscr{C}^j$ is *not* in $\mathscr{W}^{j-1}$, then

(EX) $L \in \mathscr{W}_e^j$ if $\mathbf{E}(T, \mathbf{B}_L) > C_e\, E\, \ell(L)^{2-2\delta_2}$;
(HT) $L \in \mathscr{W}_h^j$ if $L \notin \mathscr{W}_e^j$ and $\mathbf{h}(T, \mathbf{B}_L) > C_h E^{\frac{1}{2m}} \ell(L)^{1+\beta_2}$;
(NN) $L \in \mathscr{W}_n^j$ if $L \notin \mathscr{W}_e^j \cup \mathscr{W}_h^j$ but it intersects an element of $\mathscr{W}^{j-1}$;

if none of the above occurs, then $L \in \mathscr{S}^j$.

We finally set

$$\mathbf{\Gamma} := [-4,4]^m \setminus \bigcup_{L \in \mathscr{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathscr{S}^j} L. \tag{5.23}$$

Observe that, if $j > N_0$ and $L \in \mathscr{S}^j \cup \mathscr{W}^j$, then necessarily its father belongs to $\mathscr{S}^{j-1}$.

For what concerns the interpolating functions g_L, they are obtained as the result of the following procedure.

(1) Let $L \in \mathscr{S} \cup \mathscr{W}$ and π_L be an optimal plane. Then, $T \llcorner \bar{C}_{32 r_L}(p_L, \pi_L)$ fulfills the assumptions of the approximation Theorem 3.3.2 in the cylinder $\bar{C}_{32 r_L}(p_L, \pi_L)$, and we can then construct a Lipschitz approximation

$$f_L : B_{8 r_L}(p_L, \pi_L) \to \mathcal{A}_Q(\pi_L^\perp).$$

(2) We let $h_L : B_{7 r_L}(p_L, \pi_L) \to \pi_L^\perp$ be a regularization of the average given by

$$h_L := (\boldsymbol{\eta} \circ f_L) * \varrho_{\ell(L)},$$

where $\varrho \in C_c^\infty(B_1)$ is radial, $\int \varrho = 1$ and $\int |x|^2 \varrho(x)\, dx = 0$.

(3) Finally, we find a smooth map $g_L : B_{4 r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that

$$\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \bar{C}_{4 r_L}(p_L, \pi_0),$$

where we recall that $\mathbf{G}_u$ denotes the current induced by the graph of a function u.

The fact that the above procedure can be applied follows from the choice of the stopping criteria for the construction of the Whitney decomposition. We refer to [14] for a detailed proof. Here we only stress the fact that this construction depends strongly on the choice of the constants involved: in particular, $C_e, C_h, \beta_2, \delta_2, M_0$ are positive real numbers and N_0 a natural number satisfying in particular

$$\beta_2 = 4\,\delta_2 = \min\left\{\frac{1}{2m}, \frac{\gamma_1}{100}\right\}, \tag{5.24}$$

where γ_1 is the constant of Theorem 3.3.2, and

$$M_0 \geq C_0(m, n, \bar{n}, Q) \geq 4 \quad \text{and} \quad \sqrt{m} M_0 2^{7-N_0} \leq 1\,. \tag{5.25}$$

Note that β_2 and δ_2 are fixed, while the other parameters are not fixed but are subject to further restrictions in the various statements, respecting a very precise "hierarchy" (cp. [14, Assumption 1.9]).

Finally, we add also a few words concerning the construction of the normal approximation N. In every Whitney region $\mathcal{L}$ the map N is a suitable extension of the reparametrization of the Lipschitz approximation f_L. Then the estimates (5.18), (5.19) and (5.20) follow easily from Theorem 3.3.2. The most intricate proof is the one of (5.21) for which the choice of the regularization h_L deeply plays a role. The main idea is that, on the optimal plane π_L, the average of the sheets of the minimizing current is almost the graph of a harmonic function. Therefore, a good way to regularize it (which actually would keep it unchanged if it were exactly harmonic) is to convolve with a radial symmetric mollifier. This procedure, which we stress is not the only possible one, will indeed preserve the main properties of the average.

5.5. **Splitting before tilting.** The above criteria are not just important for the construction purposes, but also lead to a couple of important estimates which will be referred to as *splitting-before-tilting* estimates. Indeed, it is not a case that the powers of the side-length in the (EX) and (HT) criteria look like the powers in the familiar decay of the excess and in the height bound. In fact it turns out that, following the arguments for the height bound and for the decay of the excess, one can infer two further consequences of the Whitney decomposition's criteria.

5.5.1. *(HT)-cubes.* If a dyadic cube L has been selected by the Whitney decomposition procedure for the height criterion, then the $\mathcal{M}$-normal approximation above the corresponding Whitney region needs to have a large pointwise separation (see (5.28) below).

Proposition 5.5.2 ((HT)-estimate)**.** *If ε_2 is sufficiently small, then the following conclusions hold for every $L \in \mathscr{W}_h$:*

$$\Theta(T,p) \leq Q - \frac{1}{2} \quad \forall\, p \in \mathbf{B}_{16r_L}(p_L), \tag{5.26}$$

$$L \cap H = \emptyset \quad \forall\, H \in \mathscr{W}_n \;\; \textit{with } \ell(H) \leq \frac{1}{2}\ell(L); \tag{5.27}$$

$$\mathcal{G}\big(N(x), Q\,[\![\boldsymbol{\eta} \circ N(x)]\!]\big) \geq \frac{1}{4} C_h E^{\frac{1}{2m}} \ell(L)^{1+\beta_2} \quad \forall\, x \in \mathcal{L}. \tag{5.28}$$

A simple corollary of the previous proposition is the following.

Corollary 5.5.3. *Given any $H \in \mathscr{W}_n$ there is a chain $L = L_0, L_1, \ldots, L_j = H$ such that:*

(a) *$L_0 \in \mathscr{W}_e$ and $L_i \in \mathscr{W}_n$ for all $i = 1, \ldots, j$;*
(b) *$L_i \cap L_{i-1} \neq \emptyset$ and $\ell(L_i) = \frac{1}{2}\ell(L_{i-1})$ for all $i = 1, \ldots, j$.*

In particular, $H \subset B_{3\sqrt{m}\ell(L)}(x_L, \pi_0)$.

We use this last corollary to partition $\mathscr{W}_n$.

Definition 5.5.4 (Domains of influence)**.** We first fix an ordering of the cubes in $\mathscr{W}_e$ as $\{J_i\}_{i\in\mathbb{N}}$ so that their side-length decreases. Then $H \in \mathscr{W}_n$ belongs to $\mathscr{W}_n(J_0)$ if there is a chain as in Corollary 5.5.3 with $L_0 = J_0$. Inductively, $\mathscr{W}_n(J_r)$ is the set of cubes $H \in \mathscr{W}_n \setminus \cup_{i<r}\mathscr{W}_n(J_i)$ for which there is a chain as in Corollary 5.5.3 with $L_0 = J_r$.

5.5.5. *(Ex)-cubes.* Similarly, if a cube of the Whitney decomposition is selected by the (EX) condition, i.e. the excess does not decay at some given scale, then a certain amount of separation between the sheets of the current must also in this case occur.

Proposition 5.5.6 ((EX)-estimate)**.** *If $L \in \mathscr{W}_e$ and $\Omega = \mathbf{\Phi}(B_{\ell(L)/4}(q, \pi_0))$ for some point $q \in \pi_0$ with dist$(L, q) \leq 4\sqrt{m}\,\ell(L)$, then*

$$C_e E \ell(L)^{m+2-2\delta_2} \leq \ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C \int_\Omega |DN|^2 \,, \tag{5.29}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C\ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C\ell(L)^{-2} \int_\Omega |N|^2 \,. \tag{5.30}$$

Both propositions above are a typical *splitting-before-tilting* phenomenon in this sense: the key assumption is that the excess has decayed up to a given scale (i.e. no "tilting" occurs), while the conclusion is that a certain amount of separation between the sheets of the current ("splitting") holds. We borrowed this terminology from the paper by T. Rivière [29], where a similar phenomenon (but not completely the same) was proved for semi-calibrated two dimensional currents as a consequence of a lower epi-perimetric inequality.

5.6. **Intervals of flattening.** Here we define the last feature of the construction of the center manifold, namely the so called interval of flattening. A center manifold constitutes a good approximation of the average of the sheets of a current as soon as the errors in Theorem 5.3.2 are small compared to the distance from the origin. In this case, we are forced to interrupt our blowup analysis and to start a new center manifold. This procedure is explained in details in the following paragraph.

5.6.1. *Defining procedure.* We fix the constant $c_s := \frac{1}{64\sqrt{m}}$ and notice that $2^{-N_0} < c_s$. We set

$$\mathcal{R} := \big\{r \in]0,1] : \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon_3^2\big\}, \tag{5.31}$$

where $\varepsilon_3 > 0$ is a suitably chosen constant, always assumed to be smaller than ε_2. Observe that, if $(s_k) \subset \mathcal{R}$ and $s_k \uparrow s$, then $s \in \mathcal{R}$. We cover $\mathcal{R}$ with a collection $\mathcal{F} = \{I_j\}_j$ of intervals $I_j =]s_j, t_j]$ defined as follows: we start with

$$t_0 := \max\{t : t \in \mathcal{R}\}.$$

Next assume, by induction, to have defined

$$t_0 > s_0 \geq t_1 > s_1 \geq \ldots > s_{j-1} \geq t_j,$$

and consider the following objects:

- $T_j := ((\iota_{0,t_j})_\sharp T) \llcorner \mathbf{B}_{6\sqrt{m}}$, and assume (without loss of generality, up to a rotation) that $\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}, \pi_0) = \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})$;
- let $\mathcal{M}_j$ the corresponding center manifold for T_j, given as the graph of a map $\boldsymbol{\varphi}_j : \pi_0 \supset [-4,4]^m \to \pi_0^\perp$, (for later purposes we set $\mathbf{\Phi}_j(x) := (x, \boldsymbol{\varphi}_j(x))$).

Then, one of the following possibilities occurs:

(Stop) either there is $r \in]0,3]$ and a cube L of the Whitney decomposition $\mathscr{W}^{(j)}$ of $[-4,4]^m \subset \pi_0$ (applied to T_j) such that

$$\ell(L) \geq c_s\, r \qquad \text{and} \qquad L \cap \bar{B}_r(0,\pi_0) \neq \emptyset; \tag{5.32}$$

(Go) or there exists no radius as in (Stop).

It is possible to show that when (Stop) occurs for some r, such r is smaller than 2^{-5}. This justifies the following:

(1) in case (Go) holds, we set $s_j := 0$, i.e. $I_j :=]0,t_j]$, and end the procedure;
(2) in case (Stop) holds we let $s_j := \bar{r}\, t_j$, where $\bar{r}$ is the maximum radius satisfying (Stop). We choose then t_{j+1} as the largest element in $\mathcal{R} \cap]0,s_j]$ and proceed iteratively.

The following are easy consequences of the definition: for all $r \in]\frac{s_j}{t_j}, 3[$, it holds

$$\mathbf{E}(T_j, \mathbf{B}_r) \leq C \varepsilon_3^2\, r^{2-2\delta_2}, \tag{5.33}$$

$$\sup\{\operatorname{dist}(x,\mathcal{M}_j) : x \in \operatorname{spt}(T_j) \cap \mathbf{p}_j^{-1}(\mathcal{B}_r(p_j))\} \leq C\, (E^j)^{\frac{1}{2m}}\, r^{1+\beta_2}, \tag{5.34}$$

where $E^j := \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})$ and $\mathbf{p}_j$ denotes the nearest point projection on $\mathcal{M}_j$ defined on a neighborhood of the center manifold (for the proof we refer to [15]).

5.7. **Families of subregions.** Let $\mathcal{M}$ be a center manifold and $\mathbf{\Phi} : \pi_0 \to \mathbb{R}^{m+n}$ the paremetrizing map. Set $q := \mathbf{\Phi}(0)$ and denote by B the projection of the geodesic ball $\mathbf{p}_{\pi_0}(\mathcal{B}_r(q))$, for some $r \in (0,4)$. Since $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}} \leq C \varepsilon_2^{1/m}$ in Theorem 5.2.2, it is simple to show that B is a C^2 convex set and that the maximal curvature of ∂B is everywhere smaller than $\frac{2}{r}$. Thus, for every $z \in \partial B$ there is a ball $B_{r/2}(y) \subset B$ whose closure touches ∂B at z.

In this section we show how one can partition the cubes of the Whitney decomposition which intersect B into disjoint families which are labeled by pairs $(L, B(L))$ cube–ball enjoying different properties.

Proposition 5.7.1. *There exists a set $\mathscr{Z}$ of pairs $(L, B(L))$ with this properties:*

(i) *if $(L, B(L)) \in \mathscr{Z}$, then $L \in \mathscr{W}_e \cup \mathscr{W}_h$, the radius of $B(L)$ is $\frac{\ell(L)}{4}$, $B(L) \subset B$ and $\operatorname{dist}(B(L), \partial B) \geq \frac{\ell(L)}{4}$;*
(ii) *if the pairs $(L, B(L)), (L', B(L')) \in \mathscr{Z}$ are distinct, then L and L' are distinct and $B(L) \cap B(L') = \emptyset$;*
(iii) *the cubes $\mathscr{W}$ which intersect B are partitioned into disjoint families $\mathscr{W}(L)$ labeled by $(L, B(L)) \in \mathscr{Z}$ such that, if $H \in \mathscr{W}(L)$, then $H \subset B_{30\sqrt{m}\ell(L)}(x_L)$.*

In this way, every cube of the Whitney decomposition intersecting B can be uniquely associated to a ball $B(L) \subset B$ for some $L \in \mathscr{W}_e \cap \mathscr{W}_h$. This will

allow to transfer the estimates form the cubes of the Whitney decomposition to the ball B.

5.7.2. *Proof of Proposition 5.7.1.* We start defining appropriate families of cubes and balls.

Definition 5.7.3 (Family of cubes)**.** We first define a family $\mathcal{T}$ of cubes in the Whitney decomposition $\mathscr{W}$ as follows:

(i) $\mathcal{T}$ includes all $L \in \mathscr{W}_h \cup \mathscr{W}_e$ which intersect B;
(ii) if $L' \in \mathscr{W}_n$ intersects B and belongs to the domain of influence $\mathscr{W}_n(L)$ of the cube $L \in \mathscr{W}_e$ as in Definition 5.5.4, then $L \in \mathcal{T}$.

It is easy to see that, if r belongs to an interval of flattening, then for every $L \in \mathcal{T}$ it holds that $\ell(L) \leq 3c_s r \leq r$ and $\mathrm{dist}(L, B) \leq 3\sqrt{m}\,\ell(L)$. Therefore, we can also define the following associated balls.

Definition 5.7.4. For every $L \in \mathcal{T}$, let x_L be the center of L and:

(a) if $x_L \in \overline{B}$, we then set $s(L) := \ell(L)$ and $B^L := B_{s(L)}(x_L, \pi)$;
(b) otherwise we consider the ball $B_{r(L)}(x_L, \pi) \subset \pi$ such that its closure touches $\overline{B}$ at exactly one point $p(L)$, we set $s(L) := r(L) + \ell(L)$ and define $B^L := B_{s(L)}(x_L, \pi)$.

We proceed to select a countable family $\mathscr{T}$ of pairwise disjoint balls $\{B^L\}$. We let $S := \sup_{L\in\mathcal{T}} s(L)$ and start selecting a maximal subcollection $\mathscr{T}_1$ of pairwise disjoint balls with radii larger than $S/2$. Clearly, $\mathscr{T}_1$ is finite. In general, at the stage k, we select a maximal subcollection $\mathscr{T}_k$ of pairwise disjoint balls which do not intersect any of the previously selected balls in $\mathscr{T}_1 \cup \ldots \cup \mathscr{T}_{k-1}$ and which have radii $r \in]2^{-k}S, 2^{1-k}S]$. Finally, we set $\mathscr{T} := \bigcup_k \mathscr{T}_k$.

Definition 5.7.5 (Family of pairs cube-balls $(L, B(L)) \in \mathscr{Z}$)**.** Recalling the convexity properties of B and $\ell(L) \leq r$, it easy to see that there exist balls $B_{\ell(L)/4}(q_L, \pi) \subset B^L \cap B$ which lie at distance at least $\ell(L)/4$ from ∂B. We denote by $B(L)$ one of such balls and by $\mathscr{Z}$ the collection of pairs $(L, B(L))$ with $B^L \in \mathscr{T}$.

Next, we partition the cubes of $\mathscr{W}$ which intersect B into disjoint families $\mathscr{W}(L)$ labeled by $(L, B(L)) \in \mathscr{Z}$ in the following way. Let $H \in \mathscr{W}$ have nonempty intersection with B. Then, either H is in $\mathcal{T}$ and we set $J := H$, or is in the domain of influence of some $J \in \mathcal{T}$. If $J \neq H$, then the separation between J and H is at most $3\sqrt{m}\ell(J)$ and, hence, $H \subset B_{4\sqrt{m}\ell(J)}(x_J)$. By construction there is a $B^L \in \mathscr{T}$ with $B^J \cap B^L \neq \emptyset$ and radius $s(L) \geq \frac{s(J)}{2}$. We then prescribe $H \in \mathscr{W}(L)$. Observe that

$$s(L) \leq 4\sqrt{m}\,\ell(L) \quad \text{and} \quad s(J) \geq \ell(J).$$

Therefore, it also holds

$$\ell(J) \leq 8\sqrt{m}\,\ell(L) \quad \text{and} \quad |x_J - x_L| \leq 5s(L) \leq 20\sqrt{m}\,\ell(L),$$

thus implying

$$H \subset B_{4\sqrt{m}\,\ell(J)}(x_J) \subset B_{4\sqrt{m}\ell(J)+20\sqrt{m}\,\ell(L)}(x_L) \subset B_{30\sqrt{m}\,\ell(L)}(x_L)\,.$$

6. Order of contact

In this section we discuss the issues in steps (D) and (E) of the sketch of proof in § 2.7, i.e. the order of contact of the normal approximation with the center manifold.

The key word for this part is *frequency function*, which is the monotone quantity discovered by Almgren controlling the vanishing order of a harmonic function. In order to explain this point, we consider first the case of a real valued harmonic function $f : B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ with an expansion in polar coordinates

$$f(r,\theta) = a_0 + \sum_{k=1}^{\infty} r^k \big(a_k\,\cos(k\theta) + b_k\,\sin(k\theta)\big).$$

How can one detect the smallest index k such that a_k or b_k is not 0? It is not difficult to show that the quantity

$$I_f(r) := \frac{r\int_{B_r}|\nabla f|^2}{\int_{\partial B_r}|f|^2} \tag{6.1}$$

is monotone increasing in r and its limit as $r \downarrow 0$ gives exactly the smallest non-zero index in the expansion above.

I_f is what Almgren calls the frequency function (and the reason for such terminology is now apparent from the example above), and one of the most striking discoveries of Almgren is that the monotonicity of the frequency remains true for Q-valued functions and in fact allows to obtain a non-trivial blowup limit.

In the next subsections, we see how this discussion generalizes to the case of area minimizing currents, where an *almost monotonicity* formula can be derived for a suitable frequency defined for the $\mathcal{M}$-normal approximation.

6.1. **Frequency function's estimate.**

For every interval of flattening $I_j =]s_j, t_j]$, let N_j be the normal approximation of T_j on $\mathcal{M}_j$. Since the L^2 norm of the trace of N_j may not have any connection to the current itself (remember that N_j misses a set of positive measure of T_j), we need to introduce an averaged version of the frequency function. To this aim, consider the following piecewise linear function $\phi : [0+\infty[\to [0,1]$ given by

$$\phi(r) := \begin{cases} 1 & \text{for } r \in [0, \frac{1}{2}], \\ 2-2r & \text{for } r \in]\frac{1}{2}, 1], \\ 0 & \text{for } r \in]1, +\infty[, \end{cases}$$

and let us define a new frequency function in the following way.

Definition 6.1.1. For every $r \in]0,3]$ we define:

$$\mathbf{D}_j(r) := \int_{\mathcal{M}^j} \phi\left(\frac{d_j(p)}{r}\right) |DN_j|^2(p)\, dp,$$

and

$$\mathbf{H}_j(r) := -\int_{\mathcal{M}^j} \phi'\left(\frac{d_j(p)}{r}\right) \frac{|N_j|^2(p)}{d(p)}\, dp\,,$$

where $d_j(p)$ is the geodesic distance on $\mathcal{M}_j$ between p and $\mathbf{\Phi}_j(0)$. If we have that $\mathbf{H}_j(r) > 0$, then we define the *frequency function*

$$\mathbf{I}_j(r) := \frac{r\,\mathbf{D}_j(r)}{\mathbf{H}_j(r)}.$$

Note that, by the Coarea formula,

$$\begin{aligned}\mathbf{H}_j(r) &= 2\int_{\mathcal{B}_r\setminus\mathcal{B}_{r/2}(\Phi_j(0))} \frac{|N|^2}{d(p)}\\ &= 2\int_{r/2}^{r} \frac{1}{t}\int_{\partial\mathcal{B}_t(\Phi_j(0))} |N_j|^2\, dt\,,\end{aligned} \tag{6.2}$$

whereas, using Fubini,

$$\begin{aligned}r\,\mathbf{D}_j(r) &= \int_{\mathcal{M}_j} |DN_j|^2(x)\int_{\frac{r}{2}}^{r} \mathbf{1}_{]|x|,\infty[}(t)\, dt\, d\mathcal{H}^m(x)\\ &= 2\int_{\frac{r}{2}}^{r}\int_{\mathcal{B}_t(\mathbf{\Phi}_j(0))} |DN_j|^2\, dt.\end{aligned} \tag{6.3}$$

This explains in which sense $\mathbf{I}_j$ is an average of the quantity introduced by F. Almgren.

The main analytical estimate is then the following.

Theorem 6.1.2. *If ε_3 in (5.31) is sufficiently small, then there exists a constant $C > 0$ (indepent of j) such that, if $[a,b] \subset [\frac{s}{t}, 3]$ and $\mathbf{H}_j|_{[a,b]} > 0$, then it holds*

$$\mathbf{I}_j(a) \leq C(1 + \mathbf{I}_j(b)). \tag{6.4}$$

To simplify the notation, we drop the index j and omit the measure $\mathcal{H}^m$ in the integrals over regions of $\mathcal{M}$. For the proof of the theorem we need to introduce some auxiliary functions (all absolutely continuous with respect to r). We let $\partial_{\hat r}$ denote the derivative along geodesics starting at $\mathbf{\Phi}(0)$. We set

$$\mathbf{E}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) \sum_{i=1}^{Q} \langle N_i(p), \partial_{\hat r} N_i(p)\rangle\, dp\,,$$

$$\mathbf{G}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) d(p)\, |\partial_{\hat r} N(p)|^2\, dp,$$

$$\mathbf{\Sigma}(r) := \int_{\mathcal{M}} \phi\left(\tfrac{d(p)}{r}\right) |N|^2(p)\, dp\,.$$

The proof of Theorem 6.1.2 exploits some "integration by parts" formulas, which in our setting are given by the first variations for the minimizing current. We collect these identities in the following proposition, and proceed then with the proof of the theorem.

Proposition 6.1.3. *There exist dimensional constants $C, \gamma_3 > 0$ such that, if the hypotheses of Theorem 6.1.2 hold and $\mathbf{I} \geq 1$, then*

$$\left|\mathbf{H}'(r) - \tfrac{m-1}{r}\,\mathbf{H}(r) - \tfrac{2}{r}\,\mathbf{E}(r)\right| \leq C\mathbf{H}(r), \tag{6.5}$$

$$\left|\mathbf{D}(r) - r^{-1}\mathbf{E}(r)\right| \leq C\mathbf{D}(r)^{1+\gamma_3} + C\varepsilon_3^2\,\mathbf{\Sigma}(r), \tag{6.6}$$

$$\left|\mathbf{D}'(r) - \tfrac{m-2}{r}\,\mathbf{D}(r) - \tfrac{2}{r^2}\,\mathbf{G}(r)\right| \leq C\mathbf{D}(r) + C\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)^{1+\gamma_3}, \tag{6.7}$$

$$\mathbf{\Sigma}(r) + r\,\mathbf{\Sigma}'(r) \leq C\,r^2\,\mathbf{D}(r) \leq Cr^{2+m}\varepsilon_3^2. \tag{6.8}$$

We assume for the moment the proposition and prove the theorem.

Proof of Theorem 6.1.2. It enough to consider the case in which $\mathbf{I} > 1$ on $]a, b[$. Set $\mathbf{\Omega}(r) := \log \mathbf{I}(r)$. By Proposition 6.1.3, if ε_3 is sufficiently small, then

$$\frac{\mathbf{D}(r)}{2} \leq \frac{\mathbf{E}(r)}{r} \leq 2\,\mathbf{D}(r), \tag{6.9}$$

from which we conclude that $\mathbf{E} > 0$ over the interval $]a, b'[$. Set for simplicity $\mathbf{F}(r) := \mathbf{D}(r)^{-1} - r\mathbf{E}(r)^{-1}$, and compute

$$-\mathbf{\Omega}'(r) = \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{1}{r} \overset{(6.6)}{=} \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} - \mathbf{D}'(r)\mathbf{F}(r) - \frac{1}{r}.$$

Again by Proposition 6.1.3:

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \overset{(6.5)}{\leq} \frac{m-1}{r} + C + \frac{2}{r}\,\frac{\mathbf{E}(r)}{\mathbf{H}(r)}, \tag{6.10}$$

$$|\mathbf{F}(r)| \overset{(6.6)}{\leq} C\,\frac{r(\mathbf{D}(r)^{1+\gamma_3} + \mathbf{\Sigma}(r))}{\mathbf{D}(r)\,\mathbf{E}(r)} \overset{(6.9)}{\leq} C\,\mathbf{D}(r)^{\gamma_3 - 1} + C\,\frac{\mathbf{\Sigma}(r)}{\mathbf{D}(r)^2}, \tag{6.11}$$

$$\begin{aligned}
-\frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} &\overset{(6.7)}{\leq} \left(C - \frac{m-2}{r}\right)\frac{r\mathbf{D}(r)}{\mathbf{E}(r)} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} \\
&\quad + C\frac{r\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + \mathbf{D}(r)^{1+\gamma_3}}{\mathbf{E}(r)} \\
&\leq C - \frac{m-2}{r} + \frac{C}{r}\mathbf{D}(r)|\mathbf{F}(r)| - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} \\
&\quad + C\mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + C\frac{\mathbf{D}(r)^{\gamma_3}}{r} \\
&\overset{(6.8),(6.11)}{\leq} C - \frac{m-2}{r} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + C\mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + C\,r^{\gamma_3\,m - 1},
\end{aligned} \tag{6.12}$$

where we used the rough estimate $\mathbf{D}(r) \leq C\, r^{m+2-2\delta_2}$ coming from (5.19) of Theorem 5.3.2 and the condition (Stop).

By Cauchy-Schwartz, we have

$$\frac{\mathbf{E}(r)}{r\mathbf{H}(r)} \leq \frac{\mathbf{G}(r)}{r\mathbf{E}(r)}. \tag{6.13}$$

Thus, by (6.10), (6.12) and (6.13), we conclude

$$\begin{aligned} -\mathbf{\Omega}'(r) &\leq C + C\, r^{\gamma_3\, m-1} + Cr\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) - \mathbf{D}'(r)\mathbf{F}(r) \\ &\overset{(6.11)}{\leq} C\, r^{\gamma_3\, m-1} + C\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2}. \end{aligned} \tag{6.14}$$

Integrating (6.14) we conclude:

$$\begin{aligned} \mathbf{\Omega}(a) - \mathbf{\Omega}(b) \leq C + C\left(\mathbf{D}(b)^{\gamma_3} - \mathbf{D}(a)^{\gamma_3}\right) \\ + C\left[\frac{\mathbf{\Sigma}(a)}{\mathbf{D}(a)} - \frac{\mathbf{\Sigma}(b)}{\mathbf{D}(b)} + \int_a^b \frac{\mathbf{\Sigma}'(r)}{\mathbf{D}(r)}\, dr\right] \overset{(6.8)}{\leq} C. \end{aligned}$$

□

6.1.4. *Proof of Proposition 6.1.3.* The remaining part of this subsection is devoted to give some arguments for the proof of the first variation formulas.

The estimate (6.5) follows from a straightforward computation: using the area formula and setting $y = rz$, we have

$$\mathbf{H}(r) = -r^{m-1}\int_{T_q\mathcal{M}} \frac{\phi'\left(|z|\right)}{|z|}\, |N|^2(\exp(rz))\, \mathbf{J}\exp(rz)\, dx,$$

and differentiating under the integral sign, we easily get (6.5):

$$\begin{aligned} \mathbf{H}'(r) = &- (m-1)\, r^{m-2}\int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|}\, |N|^2(\exp(rz))\, \mathbf{J}\exp(rz)\, dz \\ &- 2\, r^{m-1}\int_{T_q\mathcal{M}} \phi'(|z|) \sum_i \langle N_i, \partial_{\hat r} N_i\rangle\, (\exp(rz))\, \mathbf{J}\exp(rz)\, dz \\ &- r^{m-1}\int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|}\, |N|^2(\exp(rz))\, \frac{d}{dr}\mathbf{J}\exp(rz)\, dz \\ = &\frac{m-1}{r}\, \mathbf{H}(r) + \frac{2}{r}\, \mathbf{E}(r) + O(1)\, \mathbf{H}(r), \end{aligned}$$

where we the following simple fact for the Jacobian of the exponential map $\frac{d}{dr}\mathbf{J}\exp(r\, z) = O(1)$, because $\mathcal{M}$ is a $C^{3,\kappa}$ submanifold and the exponential map exp is a $C^{2,\kappa}$ map.

Similarly, (6.8) follows by simple computation which involve a Poincaré inequality: namely, if $\mathbf{I} \geq 1$, then

$$\int_{\mathcal{B}_r(q)} |N|^2 \leq C\, r^2\mathbf{D}(r). \tag{6.15}$$

We refer to [15] for the details of the proof.

Here we try to explain the remaining two estimates, which instead are connected to the first variation $\delta T(X)$ of the area minimizing current T along a vector field X.

The idea is the following: since the first variations of T are zero, we compute them using its approximation N and derive the integral equality in the Proposition 6.1.3. To understand the meaning of these estimates, consider $u:\mathbb{R}^m\to\mathbb{R}^n$ a harmonic function. Then, computing the variations of the Dirichlet energy of u leads to the following two identities:

$$\int_{B_r}|Du|^2=\int_{\partial B_r}u\cdot\frac{\partial u}{\partial\nu},$$
$$\int_{\partial B_r}|Du|^2=\frac{m-2}{r}\int_{B_r}|Du|^2+2\int_{\partial B_r}\left|\frac{\partial u}{\partial\nu}\right|^2,$$

which are the exact analog of (6.6) and (6.7) without any error term. What we need to do is then to replace the Dirichlet energy with the area functional, and to consider the fact that the normal approximation N is only approximately stationary with respect to this functional.

We start fixing a tubular neighborhood $\mathbf{U}$ of $\mathcal{M}$ and the normal projection $\mathbf{p}:\mathbf{U}\to\mathcal{M}$. Observe that $\mathbf{p}\in C^{2,\kappa}$. We will consider:

(1) the *outer variations*, where $X(p)=X_o(p):=\phi\left(\frac{d(\mathbf{p}(p))}{r}\right)(p-\mathbf{p}(p))$.
(2) the *inner variations*, where $X(p)=X_i(p):=Y(\mathbf{p}(p))$ with

$$Y(p):=\frac{d(p)}{r}\,\phi\left(\frac{d(p)}{r}\right)\frac{\partial}{\partial\hat r}\quad\forall\,p\in\mathcal{M}.$$

Consider now the map $F(p):=\sum_i [\![p+N_i(p)]\!]$ and the current $\mathbf{T}_F$ associated to its image. Observe that X_i and X_o are supported in $\mathbf{p}^{-1}(\mathcal{B}_r(q))$ but none of them is *compactly* supported. However, it is simple to see that $\delta T(X)=0$. Then, we have

$$\begin{aligned}|\delta\mathbf{T}_F(X)|&=|\delta\mathbf{T}_F(X)-\delta T(X)|\\&\le\underbrace{\int_{\mathrm{spt}\,(T)\setminus\mathrm{Im}(F)}\left|\mathrm{div}_{\vec T}X\right|\,d\|T\|+\int_{\mathrm{Im}(F)\setminus\mathrm{spt}\,(T)}\left|\mathrm{div}_{\vec{\mathbf{T}}_F}X\right|\,d\|\mathbf{T}_F\|}_{\mathrm{Err}_4},\end{aligned}\tag{6.16}$$

where $\mathrm{Im}(F)$ is the image of the map $F(x)=\sum_i[\![(x,N_i(x))]\!]$, i.e. the support of the current $\mathbf{T}_F$.

Set now for simplicity $\varphi_r(p):=\phi\big(\frac{d(p)}{r}\big)$. It is not hard to realize that the mass of the current $\mathbf{T}_F$ can be expressed in the following way:

$$\begin{aligned}\mathbf{M}(\mathbf{T}_F)=Q\,\mathcal{H}^m(\mathcal{M})-Q\int_{\mathcal{M}}\langle H,\boldsymbol{\eta}\circ N\rangle+\frac12\int_{\mathcal{M}}|DN|^2\\+\int_{\mathcal{M}}\sum_i\Big(P_2(x,N_i)+P_3(x,N_i,DN_i)+R_4(x,DN_i)\Big),\end{aligned}\tag{6.17}$$

where P_2, P_3 and R_4 are quadratic, cubic and fourth order errors terms (see [16, Theorem 3.2]) One can then compute the first variation of a push-forward current $\mathbf{T}_F$ and obtain (cp. [16, Theorem 4.2])

$$\delta\mathbf{T}_F(X_o) = \int_{\mathcal{M}} \Big(\varphi_r\,|DN|^2 + \sum_{i=1}^{Q} N_i \otimes \nabla\varphi_r : DN_i\Big) + \sum_{j=1}^{3} \mathrm{Err}_j^o, \tag{6.18}$$

where the errors Err_j^o satisfy

$$\mathrm{Err}_1^o = -Q \int_{\mathcal{M}} \varphi_r \langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle, \tag{6.19}$$

$$|\mathrm{Err}_2^o| \le C \int_{\mathcal{M}} |\varphi_r||A|^2|N|^2, \tag{6.20}$$

$$|\mathrm{Err}_3^o| \le C \int_{\mathcal{M}} \big(|N||A| + |DN|^2\big)\big(|\varphi_r||DN|^2 + |D\varphi_r||DN||N|\big)\,, \tag{6.21}$$

here $H_{\mathcal{M}}$ is the mean curvature vector of $\mathcal{M}$. Plugging (6.18) into (6.16), we then conclude

$$\left|\mathbf{D}(r) - r^{-1}\mathbf{E}(r)\right| \le \sum_{j=1}^{4} \left|\mathrm{Err}_j^o\right|\,, \tag{6.22}$$

where Err_4^o corresponds to Err_4 of (6.16) when $X = X_o$. Arguing similarly with $X = X_i$ (cp. [16, Theorem 4.3]), we get

$$\delta\mathbf{T}_F(X_i) = \frac{1}{2}\int_{\mathcal{M}} \Big(|DN|^2 \mathrm{div}_{\mathcal{M}} Y - 2\sum_{i=1}^{Q}\langle DN_i : (DN_i \cdot D_{\mathcal{M}} Y)\rangle\Big) + \sum_{j=1}^{3} \mathrm{Err}_j^i\,, \tag{6.23}$$

where this time the errors Err_j^i satisfy

$$\mathrm{Err}_1^i = -Q \int_{\mathcal{M}} \big(\langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\,\mathrm{div}_{\mathcal{M}} Y + \langle D_Y H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\big)\,, \tag{6.24}$$

$$|\mathrm{Err}_2^i| \le C \int_{\mathcal{M}} |A|^2 \big(|DY||N|^2 + |Y||N|\,|DN|\big)\,, \tag{6.25}$$

$$\begin{aligned}|\mathrm{Err}_3^i| \le C \int_{\mathcal{M}} &|Y||A||DN|^2\big(|N| + |DN|\big)\\ &+ |DY|\big(|A|\,|N|^2|DN| + |DN|^4\big)\Big)\,.\end{aligned} \tag{6.26}$$

Straightforward computations lead to

$$D_{\mathcal{M}} Y(p) = \phi'\left(\frac{d(p)}{r}\right)\frac{d(p)}{r^2}\frac{\partial}{\partial\hat r}\otimes\frac{\partial}{\partial\hat r} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{\mathrm{Id}}{r} + O(1)\right)\,, \tag{6.27}$$

$$\mathrm{div}_{\mathcal{M}}\, Y(p) = \phi'\left(\frac{d(p)}{r}\right)\frac{d(p)}{r^2} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{m}{r} + O(1)\right)\,. \tag{6.28}$$

Plugging (6.27) and (6.28) into (6.23) and using (6.16) we then conclude

$$\left|\mathbf{D}'(r) - (m-2)r^{-1}\mathbf{D}(r) - 2r^{-2}\mathbf{G}(r)\right| \leq C\mathbf{D}(r) + \sum_{j=1}^{4}\left|\mathrm{Err}_j^i\right| . \tag{6.29}$$

Proposition 6.1.3 is then proved by the estimates of the errors terms done in the next subsection.

6.1.5. *Estimates of the errors terms.* We consider the family of pairs $\mathscr{Z} = \{(J_i, B(J_i))\}_{i\in\mathbb{N}}$ introduced in the previous section, and set

$$\mathcal{B}^i := \mathbf{\Phi}(B(J_i)) \quad \text{and} \quad \mathcal{U}_i = \cup_{H\in\mathscr{W}(J_i)}\mathbf{\Phi}(H)\cap\mathcal{B}_r(q)\,.$$

Set $\mathcal{V}_i := \mathcal{U}_i\setminus\mathcal{K}$, where $\mathcal{K}$ is the coincidence set of Theorem 5.3.2. By a simple application of Theorem 5.3.2 we derive the following estimates:

$$\int_{\mathcal{U}_i}|\boldsymbol{\eta}\circ N| \leq C E\,\ell_i^{2+m+\frac{\gamma_2}{2}} + C\int_{\mathcal{U}^i}|N|^{2+\gamma_2}, \tag{6.30}$$

$$\int_{\mathcal{U}_i}|DN|^2 \leq C E\,\ell_i^{m+2-2\delta_2}, \tag{6.31}$$

$$\|N\|_{C^0(\mathcal{U}_i)} + \sup_{p\in\mathrm{spt}\,(T)\cap\mathbf{p}^{-1}(\mathcal{U}_i)}|p-\mathbf{p}(p)| \leq C E^{\frac{1}{2m}}\ell_i^{1+\beta_2}, \tag{6.32}$$

$$\mathrm{Lip}(N|_{\mathcal{U}_i}) \leq C E^{\gamma_2}\ell_i^{\gamma_2}, \tag{6.33}$$

$$\mathbf{M}(T\llcorner\mathbf{p}^{-1}(\mathcal{V}_i)) + \mathbf{M}(\mathbf{T}_F\llcorner\mathbf{p}^{-1}(\mathcal{V}_i)) \leq C E^{1+\gamma_2}\ell_i^{m+2+\gamma_2}. \tag{6.34}$$

Observe that the separation between $\mathcal{B}^i$ and $\partial\mathcal{B}_r(q)$ is larger than $\ell(J_i)/4$ by Proposition 5.7.1 (i), and then $\varphi_r(p) = \phi\big(\frac{d(p)}{r}\big)$ satisfies

$$\inf_{p\in\mathcal{B}^i}\varphi_r(p) \geq (4r)^{-1}\ell_i\,, \tag{6.35}$$

where $\ell_i := \ell(J_i)$. From this and Proposition 5.7.1 (iii), we also obtain

$$\sup_{p\in\mathcal{U}_i}\varphi_r(p) - \inf_{p\in\mathcal{U}_i}\varphi_r(p) \leq C\mathrm{Lip}(\varphi_r)\ell_i \leq \frac{C}{r}\,\ell_i \stackrel{(6.35)}{\leq} C\inf_{p\in\mathcal{B}_i}\varphi_r(p)\,,$$

which translates into

$$\sup_{p\in\mathcal{U}_i}\varphi_r(p) \leq C\inf_{p\in\mathcal{B}^i}\varphi_r(p)\,. \tag{6.36}$$

Moreover, by an application of the *splitting-before-tilting* estimates in Proposition 5.5.2 and Proposition 5.5.6, we infer that

$$\int_{\mathcal{B}^i}|N|^2 \geq c\,E^{\frac{1}{m}}\ell_i^{m+2+2\beta_2} \quad \text{if } L_i\in\mathscr{W}_h, \tag{6.37}$$

$$\int_{\mathcal{B}^i}|DN|^2 \geq c\,E\,\ell_i^{m+2-2\delta_2} \quad \text{if } L_i\in\mathscr{W}_e\,. \tag{6.38}$$

This easily implies the following estimates under the hypotheses $\mathbf{I} \geq 1$: by applying (6.15), (6.35), (6.37) and (6.38), we get, for suitably chosen $\gamma(t), C(t) > 0$,

$$\begin{aligned}\sup_i E^t \Big[\ell_i^t + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)^{\frac{t}{2}} \ell_i^{\frac{t}{2}}\Big] &\leq C(t) \sup_i \Big(\int_{\mathcal{B}^i} \varphi_r(|DN|^2 + |N|^2)\Big)^{\gamma(t)} \\ &\leq C(t)\mathbf{D}(r)^{\gamma(t)}\,, \end{aligned} \tag{6.39}$$

and similarly

$$\begin{aligned}\sum_i \big(\inf_{\mathcal{B}^i} \varphi_r\big) E\, \ell_i^{m+2+\frac{\gamma_2}{4}} &\leq C \sum_i \int_{\mathcal{B}^i} \varphi_r(|DN|^2 + |N|^2) \\ &\leq C\mathbf{D}(r)\,, \end{aligned} \tag{6.40}$$

$$\begin{aligned}\sum_i E\, \ell_i^{m+2+\frac{\gamma_2}{4}} &\leq C \int_{\mathcal{B}_r(q)} \big(|DN|^2 + |N|^2\big) \\ &\leq C\big(\mathbf{D}(r) + r\mathbf{D}'(r)\big). \end{aligned} \tag{6.41}$$

We can now pass to estimate the errors terms in (6.6) and (6.7) in order to conclude the proof of Proposition 6.1.3.

Errors of type 1. By Theorem 5.2.2, the map $\boldsymbol{\varphi}$ defining the center manifold satisfies $\|D\boldsymbol{\varphi}\|_{C^{2,\kappa}} \leq C\, E^{\frac{1}{2}}$, which in turn implies $\|H_{\mathcal{M}}\|_{L^\infty} + \|DH_{\mathcal{M}}\|_{L^\infty} \leq C\, E^{\frac{1}{2}}$ (recall that $H_{\mathcal{M}}$ denotes the mean curvature of $\mathcal{M}$). Therefore, by (6.36), (6.30), (6.40) and (6.39), we get

$$\begin{aligned}|\mathrm{Err}_1^o| &\leq C \int_{\mathcal{M}} \varphi_r\, |H_{\mathcal{M}}|\, |\boldsymbol{\eta} \circ N| \\ &\leq C\, E^{\frac{1}{2}} \sum_j \Big(\big(\sup_{\mathcal{U}_i} \varphi_r\big)\, E\, \ell_j^{2+m+\gamma_2} + C \int_{\mathcal{U}_j} \varphi_r |N|^{2+\gamma_2}\Big) \\ &\leq C\mathbf{D}(r)^{1+\gamma_3} + C \sum_j E^{\frac{1}{2}}\, \ell_j^{\gamma_2(1+\beta_2)} \int_{\mathcal{U}_j} \varphi_r |N|^2 \leq C\mathbf{D}(r)^{1+\gamma_3}\,, \end{aligned}$$

and analogously

$$\begin{aligned}\big|\mathrm{Err}_1^i\big| &\leq C\, r^{-1} \int_{\mathcal{M}} \big(|H_{\mathcal{M}}| + |D_Y H_{\mathcal{M}}|\big) |\boldsymbol{\eta} \circ N| \\ &\leq C\, r^{-1} E^{\frac{1}{2}} \sum_j \Big(E\, \ell_j^{2+m+\gamma_2} + C \int_{\mathcal{U}_j} |N|^{2+\gamma_2}\Big) \\ &\leq C\, r^{-1} \mathbf{D}(r)^{\gamma} \big(\mathbf{D}(r) + r\, \mathbf{D}'(r)\big). \end{aligned}$$

Errors of type 2. From $\|A\|_{C^0} \leq C\|D\boldsymbol{\varphi}\|_{C^2} \leq CE^{\frac{1}{2}} \leq C\varepsilon_3$, it follows that $\mathrm{Err}_2^o \leq C\varepsilon_3^2 \mathbf{\Sigma}(r)$. Moreover, since $|DX_i| \leq Cr^{-1}$, (6.15) leads to

$$\big|\mathrm{Err}_2^i\big| \leq Cr^{-1} \int_{\mathcal{B}_r(p_0)} |N|^2 + C \int \varphi_r |N||DN| \leq C\mathbf{D}(r)\,.$$

Errors of type 3. Clearly, we have

$$|\mathrm{Err}_3^o| \leq \underbrace{\int \varphi_r \left(|DN|^2|N| + |DN|^4\right)}_{I_1} + C\, \underbrace{r^{-1}\int_{\mathcal{B}_r(q)} |DN|^3|N|}_{I_2}$$
$$+ C\, \underbrace{r^{-1}\int_{\mathcal{B}_r(q)} |DN||N|^2}_{I_3}\,.$$

We estimate separately the three terms (recall that $\gamma_2 > 4\delta_2$):

$$I_1 \leq \int_{\mathcal{B}_r(p_0)} \varphi_r(|N|^2|DN| + |DN|^3) \leq I_3 + C\sum_j \sup_{\mathcal{U}_j} \varphi_r E^{1+\gamma_2}\ell_j^{m+2+\frac{\gamma_2}{2}}$$
$$\stackrel{(6.40)\,\&\,(6.39)}{\leq} I_3 + C\mathbf{D}(r)^{1+\gamma_3},$$

$$I_2 \leq Cr^{-1}\sum_j E^{1+\frac{1}{2m}+\gamma_2}\ell_j^{m+3+\beta_2+\frac{\gamma_2}{2}}$$
$$\stackrel{(6.36)}{\leq} C\sum_j E^{1+\frac{1}{2m}+\gamma_2}\ell_j^{m+2+\beta_2+\frac{\gamma_2}{2}} \inf_{\mathcal{B}^j}\varphi_r \stackrel{(6.40)\,\&\,(6.39)}{\leq} C\mathbf{D}(r)^{1+\gamma_3},$$

$$I_3 \leq Cr^{-1}\sum_j E^{\gamma_2}\ell_j^{\gamma_2}\int_{\mathcal{U}_j}|N|^2 \stackrel{(6.39)}{\leq} Cr^{-1}\mathbf{D}(r)^{\gamma_3}\int_{\mathcal{B}_r(q)}|N|^2 \stackrel{(6.15)}{\leq} C\mathbf{D}(r)^{1+\gamma_3}$$

For what concerns the inner variations, we have

$$|\mathrm{Err}_3^i| \leq C\int_{\mathcal{B}_r(q)} \left(r^{-1}|DN|^3 + r^{-1}|DN|^2|N| + r^{-1}|DN||N|^2\right).$$

The last integrand corresponds to I_3, while the remaining part can be estimated as follows:

$$\int_{\mathcal{B}_r(q)} r^{-1}(|DN|^3 + |DN|^2|N|) \leq C\sum_j r^{-1}(E^{\gamma_2}\ell_j^{\gamma_2} + E^{\frac{1}{2m}}\ell_j^{1+\beta_2})\int_{\mathcal{U}_j}|DN|^2$$
$$\stackrel{(6.39)}{\leq} C\, r^{-1}\mathbf{D}(r)^{\gamma_3}\int_{\mathcal{B}_r(q)}|DN|^2$$
$$\leq C\mathbf{D}(r)^{\gamma_3}\left(\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)\right).$$

Errors of type 4. We compute explicitly

$$|DX_o(p)| \leq 2\,|p-\mathbf{p}(p)|\,\frac{|Dd(\mathbf{p}(p),q)|}{r} + \varphi_r(p)\,|D(p-\mathbf{p}(p))|$$
$$\leq C\left(\frac{|p-\mathbf{p}(p)|}{r} + \varphi_r(p)\right).$$

It follows readily from (6.16), (6.32) and (6.34) that

$$
\begin{aligned}
|\mathrm{Err}_4^o| &\le \sum_i C\Big(r^{-1}E^{\frac{1}{2m}}\ell_i^{1+\beta_2} + \sup_{\mathcal{U}_i}\varphi_r\Big)E^{1+\gamma_2}\ell_i^{m+2+\gamma_2}\\
&\overset{(6.35)\,\&\,(6.36)}{\le} C\sum_i \left[E^{\gamma_2}\ell_i^{\frac{\gamma_2}{4}}\right]\inf_{\mathcal{B}_i}\varphi_r\, E\,\ell_i^{m+2+\frac{\gamma_2}{4}} \overset{(6.40)\,\&(6.39)}{\le} C\mathbf{D}(r)^{1+\gamma_3}.
\end{aligned}
\tag{6.42}
$$

Similarly, since $|DX_i| \le Cr^{-1}$, we get

$$
\begin{aligned}
\mathrm{Err}_4^i &\le Cr^{-1}\sum_j \Big(E^{\gamma_2}\ell_j^{\frac{\gamma_2}{2}}\Big)E\,\ell_j^{m+2+\frac{\gamma_2}{2}}\\
&\overset{(6.41)\,\&\,(6.39)}{\le} C\mathbf{D}(r)^{\gamma}\left(\mathbf{D}'(r)+r^{-1}\mathbf{D}(r)\right).
\end{aligned}
$$

Remark 6.1.6. Note that the "superlinear" character of the estimates in Theorem 5.3.2 has played a fundamental role in the control of the errors.

6.2. **Boundness of the frequency.** We have proven in the previous subsection that the frequency of the $\mathcal{M}$-normal approximation remains bounded within a center manifold in the corresponding interval of flattening. In order to pass into the limit along the different center manifolds, we need also to show that the frequency remains bounded in passing from one to the other. This is again a consequence of the *splitting-before-tilting* estimates and we provide here some details of the proof, referring to [14] for the complete argument.

To simplify the notation, we set $p_j := \mathbf{\Phi}_j(0)$ and write simply $\mathcal{B}_\rho$ in place of $\mathcal{B}_\rho(p_j)$.

Theorem 6.2.1 (Boundedness of the frequency functions)**.** *If the intervals of flattening are infinitely many, then there is a number $j_0 \in \mathbb{N}$ such that*

$$
\mathbf{H}_j > 0 \text{ on }]\tfrac{s_j}{t_j}, 3[\text{ for all } j \ge j_0 \quad\text{and}\quad \sup_{j\ge j_0}\sup_{r\in]\frac{s_j}{t_j},3[}\mathbf{I}_j(r) < \infty\,. \tag{6.43}
$$

Sketch of the proof. We partition the extrema t_j of the intervals of flattening into two different classes:

(A) those such that $t_j = s_{j-1}$;
(B) those such that $t_j < s_{j-1}$.

If t_j belongs to (A), set $r := \frac{s_{j-1}}{t_{j-1}}$. Let $L \in \mathscr{W}^{(j-1)}$ be a cube of the Whitney decomposition such that $c_s\, r \le \ell(L)$ and $L \cap \bar{B}_r(0,\pi) \neq \emptyset$. Since this cube of the Whitney decomposition at step $j-1$ has size comparable with the distance to the origin, and the next center manifold starts at a comparable radius, the splitting property of the normal approximation needs to hold also for the new approximation: namely, one can show that there exists a constant $\bar{c}_s > 0$ such that

$$
\int_{\mathbf{B}_2\cap\mathcal{M}_j}|N_j|^2 \ge \bar{c}_s E^j := \mathbf{E}(T_j,\mathbf{B}_{6\sqrt{m}}),
$$

which obviously gives $\mathbf{H}_{N_j}(3) \geq cE^j$, and than $\mathbf{I}_{N_j}(3)$ is smaller than a given constant, independent of j, thus proving the theorem.

In the case t_j belongs to the class (B), then, by construction there is $\eta_j \in]0,1[$ such that $\mathbf{E}((\iota_{0,t_j})_\sharp T, \mathbf{B}_{6\sqrt{m}(1+\eta_j)}) > \varepsilon_3^2$. Up to extracting a subsequence, we can assume that $(\iota_{0,t_j})_\sharp T$ converges to a cone S: the convergence is strong enough to conclude that the excess of the cone is the limit of the excesses of the sequence. Moreover (since S is a cone), the excess $\mathbf{E}(S, \mathbf{B}_r)$ is independent of r. We then conclude

$$\varepsilon_3^2 \leq \liminf_{j\to\infty, j\in(B)} \mathbf{E}(T_j, \mathbf{B}_3)\,.$$

Thus, it follows again from the splitting phenomenon (see for details [15, Lemma 5.2]) that $\liminf_{j\to\infty, j\in(B)} \mathbf{H}_{N_j}(3) > 0$. Since $\mathbf{D}_{N_j}(3) \leq CE^j \leq C\varepsilon_3^2$, we achieve that $\limsup_{j\to\infty, j\in(B)} \mathbf{I}_{N_j}(3) > 0$, and conclude as before. □

7. Final blowup argument

We are now ready for the conclusion of the blowup argument, i.e. for the discussion of steps (F) and (G) of § 2.7.

To this aim we recall here the main results obtained so far.

We start with an m-dimensional area minimizing integer rectifiable T in $\mathbb{R}^{m+n}$ with $\partial T = 0$ and $0 \in \mathrm{D}_Q(T)$, such that there exists a sequence of radii $r_k \downarrow 0$ satisfying

$$\lim_{k\to+\infty} \mathbf{E}(T_{0,r_k}, \mathbf{B}_{10}) = 0, \tag{7.1}$$

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{0,r_k}) \cap \mathbf{B}_1) > \eta > 0, \tag{7.2}$$

$$\mathcal{H}^m\big((\mathbf{B}_1 \cap \mathrm{spt}\,(T_{0,r_k})) \setminus \mathrm{D}_Q(T_{0,r_k})\big) > 0 \quad \forall\, k \in \mathbb{N}, \tag{7.3}$$

for some constant $\alpha, \eta > 0$. In the process of solving the centering problem for such currents we have obtained the following:

(1) the intervals of flattening $I_j =]s_j, t_j]$,
(2) the center manifolds $\mathcal{M}_j$,
(3) the $\mathcal{M}_j$-normal approximations $N_j : \mathcal{M}_j \to \mathcal{A}_Q(\mathbb{R}^{m+n})$,

satisfying the conclusions of Theorem 5.2.2 and Theorem 5.3.2. It follows from the very definition of intervals of flattening that each r_k has to belong to one of these intervals. Therefore, in order to fix the ideas and to simplify the notation, we will in the sequel assume that there are infinitely many intervals of flattening and that $r_k \in I_k$: note that this is not a serious restriction, and everything holds true also in the case of finitely many intervals of flattening.

By the analysis of the order of contact and the estimate on the frequency function, see Theorem 6.1.2 and Theorem 6.2.1, we have also derived the information

$$\sup_{j\in\mathbb{N}} \sup_{r\in\left]\frac{s_j}{t_j},3\right]} \mathbf{I}_j(r) < +\infty. \tag{7.4}$$

The ultimate consequence of this estimate, thus clarifying the discussion about the non-triviality of the blowup process, is the following proposition.

Proposition 7.0.2 (Reverse Sobolev)**.** *There exists a constant $C>0$ with this property: for every $j\in\mathbb{N}$, there exists $\theta_j \in \left]\frac{3\,r_j}{2\,t_j}, 3\frac{r_j}{t_j}\right[$ such that*

$$\int_{\mathcal{B}_{\theta_j}(\mathbf{\Phi}_j(0))} |DN_j|^2 \leq C\left(\frac{t_j}{r_j}\right)^2 \int_{\mathcal{B}_{\theta_j}(\mathbf{\Phi}_j(0))} |N_j|^2 \,. \tag{7.5}$$

Proof. Set for simplicity $r:=\frac{r_j}{t_j}$ and drop the subscript $_j$ in the sequel. Using (6.2), (6.3) and (7.4), there exists $C>0$ such that

$$\begin{aligned}\int_{\frac{3}{2}r}^{3r} dt \int_{\mathcal{B}_t(\mathbf{\Phi}(0))} |DN|^2 &= \frac{3}{2} r\, \mathbf{D}(3r) \leq C\, \mathbf{H}(3r)\\ &= C\int_{\frac{3}{2}r}^{3r} dt \frac{1}{t}\int_{\partial\mathcal{B}_t(\mathbf{\Phi}(0))} |N|^2\,.\end{aligned}$$

Therefore, there must be $\theta\in[\frac{3}{2}r, 3r]$ satisfying

$$\int_{\mathcal{B}_\theta(\mathbf{\Phi}(0))} |DN|^2 \leq \frac{C}{\theta}\int_{\partial\mathcal{B}_\theta(\mathbf{\Phi}(0))} |N|^2\,. \tag{7.6}$$

This is almost the desired estimate. In oder to replace the boundary integral with a bulk integral in the right hand side of (7.6), we argue by integrating along radii in a similar way to the case of single valued functions. Fix indeed any $\sigma \in]\theta/2, \theta[$ and any point $x\in\partial\mathcal{B}_\theta(\Phi(0))$. Consider the geodesic line γ passing through x and $\Phi(0)$, and let $\hat\gamma$ be the arc on γ having one endpoint $\bar x$ in $\partial\mathcal{B}_\sigma(\Phi(0))$ and one endpoint equal to x. Using [13, Proposition 2.1(b)] and the fundamental theorem of calculus, we easily conclude

$$|N(x)| \leq |N(\bar x)| + \int_{\hat\gamma} |DN||N|\,.$$

Integrating this inequality in x and recalling that $\sigma > s/2$ we then easily conclude

$$\int_{\partial\mathcal{B}_\theta(\Phi(0))} |N|^2 \leq C\int_{\partial\mathcal{B}_\sigma(\Phi(0))} |N|^2 + C\int_{\mathcal{B}_\theta(\Phi(0))} |N|\,|DN|\,.$$

We further integrate in σ between $s/2$ and s to achieve

$$\begin{aligned}\theta\int_{\partial\mathcal{B}_\theta(\mathbf{\Phi}(0))} |N|^2 &\leq C\int_{\mathcal{B}_\theta(\mathbf{\Phi}(0))} \big(|N|^2 + \theta\,|N|\,|DN|\big)\\ &\leq \frac{\theta^2}{2\,C}\int_{\mathcal{B}_\theta(\mathbf{\Phi}(0))} |DN|^2 + C\int_{\mathcal{B}_\theta(\mathbf{\Phi}(0))} |N|^2\,.\end{aligned} \tag{7.7}$$

Combining (7.7) with (7.6) we easily conclude (7.5). ∎

7.1. **Convergence to a Dir-minimizer.** We can now define the final blowup sequence, because the Reverse Sobolev inequality proven in Proposition 7.0.2 gives the right radius θ_k for assuring compactness of the corresponding maps. To this aim set $\bar r_k := \frac{2}{3}\theta_k\, t_k \in [r_k, 2\, r_k]$, and rescale the current and the maps accordingly:

$$\bar T_k := (\iota_{0,\bar r_k})_\sharp T \quad \text{and} \quad \bar{\mathcal{M}}_k := \iota_{0,\bar r_k/t_k}\mathcal{M}_k,$$

and $\bar N_k : \bar{\mathcal{M}}_k \to \mathbb{R}^{m+n}$ for the rescaled $\bar{\mathcal{M}}_k$-normal approximations given by

$$\bar N_k(p) := \frac{t_k}{\bar r_k} N_k\left(\frac{\bar r_k p}{t_k}\right).$$

Note that the ball $\mathcal{B}_{s_k} \subset \mathcal{M}_k$ is sent into the ball $\mathcal{B}_{\frac{3}{2}} \subset \bar{\mathcal{M}}_k$. Moreover, via some elementary regularity theory of area minimizing currents, one deduces that

(1) $\mathbf{E}(\bar T_k, \mathbf{B}_{\frac{1}{2}}) \leq C\mathbf{E}(T, \mathbf{B}_{r_k}) \to 0$;
(2) $\bar T_k$ locally converge (and in the Hausdorff sense for what concerns the supports) to an m-plane with multiplicity Q;
(3) $\bar{\mathcal{M}}_k$ locally converge to the flat m-plane (without loss of generality π_0);
(4) recalling (7.2),
$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(\bar T_k) \cap \mathbf{B}_1) \geq \eta' > 0\,, \tag{7.8}$$
for some positive constant η'.

We can then consider the following definition for the blow-up maps

$$N_k^b : B_3 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{m+n})$$

given by

$$N_k^b(x) := \mathbf{h}_k^{-1}\bar N_k(\mathbf{e}_k(x))\,, \quad \text{with } \mathbf{h}_k := \|\bar N_k\|_{L^2(\mathcal{B}_{\frac{3}{2}})}, \tag{7.9}$$

where $\mathbf{e}_k : B_3 \subset \mathbb{R}^m \simeq T_{\bar p_k}\bar{\mathcal{M}}_k \to \bar{\mathcal{M}}_k$ denotes the exponential map at $\bar p_k = t_k\, \mathbf{\Phi}_k(0)/\bar r_k$.

Proposition 7.0.2 implies then that there exists a constant $C > 0$ such that, for every k,

$$\int_{B_{\frac{3}{2}}} |DN_k^b|^2 \leq C. \tag{7.10}$$

Moreover, as a simple consequence of Theorem 5.3.2 (details left to the readers), we find an exponent $\gamma > 0$ such that

$$\mathrm{Lip}(\bar N_k) \leq C\mathbf{h}_k^\gamma, \tag{7.11}$$

$$\mathbf{M}((\mathbf{T}_{\bar F_k} - \bar T_k)\llcorner(\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}})) \leq C\mathbf{h}_k^{2+2\gamma}, \tag{7.12}$$

$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta} \circ \bar N_k| \leq C\mathbf{h}_k^2\,. \tag{7.13}$$

It then follows from (7.10), $\|N_k^b\|_{L^2(\mathcal{B}_{3/3})} \equiv 1$ and the Sobolev embedding for Q-valued functions (cp. [13, Proposition 2.11]) that up to subsequences (as usual not relabeled) there exists a Sobolev function $N_\infty^b : B_{\frac{3}{2}} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ such that the maps N_k^b converge strongly in $L^2(B_{\frac{3}{2}})$ to N_∞^b. Then from (7.13) we deduce also that

$$\boldsymbol{\eta} \circ N_\infty^b \equiv 0 \quad \text{and} \quad \|N_k^b\|_{L^2(\mathcal{B}_{3/3})} \equiv 1. \tag{7.14}$$

Moreover, since the $\bar{N}_k$ are $\bar{\mathcal{M}}_k$-normal approximations and the $\bar{\mathcal{M}}_k$ converging to the flat m-dimensional plane $\mathbb{R}^m \times \{0\}$, N_∞^b takes values in the space of Q-points of $\{0\} \times \mathbb{R}^n$ (in place of the full $\mathbb{R}^{m+n}$).

To conclude our contradiction argument, we need to prove the N_∞^b is Dir-minimizing.

7.1.1. *N_∞^b is Dir-minimizing.* Apart from the necessary technicalities, the proof of this claim is very intuitive and relies on the following observation: if the energy of N_∞^b could be decreased, then one would be able to find a rectifiable current with less mass then $\bar{T}_k$, because the rescaling of N_k^b are done in terms of the L^2 norm $\mathbf{h}_k$ whereas the errors in the normal approximation are superlinear with $\mathbf{h}_k$.

Next we give all the details for this arguments.

We can consider for every $\bar{\mathcal{M}}_k$ an orthonormal frame of $(T\bar{\mathcal{M}}_k)^\perp$,

$$\nu_1^k, \ldots, \nu_n^k,$$

with the property (cf. [16, Lemma A.1]) that

$$\nu_j^k \to e_{m+j} \quad \text{in } C^{2,\kappa/2}(\bar{\mathcal{M}}_k) \text{ as } k \uparrow \infty \text{ for every } j$$

(here $e_1, \ldots, e_{m+n}$ is the standard basis of $\mathbb{R}^{m+n}$).

Given now any Q-valued map $u = \sum_i [\![u_i]\!] : \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^n)$, we can consider the map

$$\mathbf{u}_k : x \mapsto \sum_i \left[\!\!\left[(u_i(x))^j \nu_j^k(x) \right]\!\!\right],$$

where we set $(u_i)^j := \langle u_i(x), e_{m+j} \rangle$ and we use Einstein's convention. Then, the differential map $D\mathbf{u}_k := \sum_i [\![D(\mathbf{u}_k)_i]\!]$ is given by

$$D(\mathbf{u}_k)_i = D(u_i)^j \nu_j^k + (u_i)^j D\nu_j^k.$$

Taking into account that $\|D\nu_i^k\|_{C^0} \to 0$ as $k \to +\infty$, we deduce that

$$\left| \int \left(|D\mathbf{u}_k|^2 - |Du|^2 \right) \right| \le o(1) \int \left(|u|^2 + |Du|^2 \right). \tag{7.15}$$

Note that N_k^b has also the form $\mathbf{u}_k^b$ for some Q-valued function $u_k^b : \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^n)$.

We now show the Dir-minimizing property of N^b_∞. There is nothing to prove if its Dirichlet energy vanishes. We can therefore assume that there exists $c_0 > 0$ such that

$$c_0 \mathbf{h}_k^2 \leq \int_{\mathcal{B}_{\frac{3}{2}}} |D\bar{N}_k|^2 \,. \tag{7.16}$$

We argue by contradiction and assume there is a radius $t \in \left]\frac{5}{4}, \frac{3}{2}\right[$ and a function $f : B_{\frac{3}{2}} \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^n)$ such that

$$f|_{B_{\frac{3}{2}} \setminus B_t} = N^b_\infty|_{B_{\frac{3}{2}} \setminus B_t} \quad \text{and} \quad \mathrm{Dir}(f, B_t) \leq \mathrm{Dir}(N^b_\infty, B_t) - 2\,\delta,$$

for some $\delta > 0$.

Using f as a model, we need to find a sequence of functions v^b_k such that they have the same boundary data of N^b_k and less energy. This can be done because of the strong convergence of the traces and the possibility to make an interpolation between two functions with close by traces. This is one of the instances where thinking to multiple valued functions as classical single valued ones may be useful. In any case, the details are given in [17, Proposition 3.5] and lead to competitor functions v^b_k such that, for k large enough,

$$v^b_k|_{\partial B_r} = N^b_k|_{\partial B_r}, \quad \mathrm{Lip}(v^b_k) \leq C\mathbf{h}_k^\gamma,$$
$$\int_{B_{\frac{3}{2}}} |\boldsymbol{\eta} \circ v^b_k| \leq C\mathbf{h}_k^2 \quad \text{and} \quad \int_{B_{\frac{3}{2}}} |Dv^b_k|^2 \leq \int |DN^b_k|^2 - \delta\,\mathbf{h}_k^2\,,$$

where $C > 0$ is a constant independent of k. Clearly, setting $\tilde{N}_k = v^b_k \mathbf{e}_k^{-1}$ satisfy

$$\tilde{N}_k \equiv \bar{N}_k \quad \text{in } \mathcal{B}_{\frac{3}{2}} \setminus \mathcal{B}_t, \quad \mathrm{Lip}(\tilde{N}_k) \leq C\mathbf{h}_k^\gamma,$$
$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta} \circ \tilde{N}_k| \leq C\mathbf{h}_k^2 \quad \text{and} \quad \int_{\mathcal{B}_{\frac{3}{2}}} |D\tilde{N}_k|^2 \leq \int_{\mathcal{B}_{\frac{3}{2}}} |D\bar{N}_k|^2 - \delta\mathbf{h}_k^2.$$

Consider finally the map $\tilde{F}_k(x) = \sum_i [\![x + \tilde{N}_i(x)]\!]$. The current $\mathbf{T}_{\tilde{F}_k}$ coincides with $\mathbf{T}_{\bar{F}_k}$ on $\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}} \setminus \mathcal{B}_t)$. Define the function $\varphi_k(p) = \mathrm{dist}_{\bar{\mathcal{M}}_k}(0, \mathbf{p}_k(p))$ and consider for each $s \in \left]t, \frac{3}{2}\right[$ the slices $\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, s\rangle$. By (7.12) we have

$$\int_t^{\frac{3}{2}} \mathbf{M}(\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, s\rangle) \leq C\mathbf{h}_k^{2+\gamma}\,.$$

Thus we can find for each k a radius $\sigma_k \in \left]t, \frac{3}{2}\right[$ on which $\mathbf{M}(\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, \sigma_k\rangle) \leq C\mathbf{h}_k^{2+\gamma}$. By the isoperimetric inequality (see [17, Remark 4.3]) there is a current S_k such that

$$\partial S_k = \langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, \sigma_k\rangle, \quad \mathbf{M}(S_k) \leq C\mathbf{h}_k^{(2+\gamma)m/(m-1)}.$$

Our competitor current is, then, given by

$$Z_k := \bar{T}_k \llcorner (\mathbf{p}_k^{-1}(\bar{\mathcal{M}}_k \setminus \mathcal{B}_{\sigma_k})) + S_k + \mathbf{T}_{\tilde{F}_k} \llcorner (\mathbf{p}_k^{-1}(\mathcal{B}_{\sigma_k})).$$

Note that Z_k has the same boundary as $\bar{T}_k$. On the other hand, by (7.12) and the bound on $\mathbf{M}(S_k)$, we have

$$\mathbf{M}(\tilde{T}_k) - \mathbf{M}(\bar{T}_k) \le \mathbf{M}(\mathbf{T}_{\bar{F}_k}) - \mathbf{M}(\mathbf{T}_{\tilde{F}_k}) + C\mathbf{h}_k^{2+2\gamma}. \tag{7.17}$$

Denote by A_k and by H_k respectively the second fundamental forms and mean curvatures of the manifolds $\bar{\mathcal{M}}_k$. Using the Taylor expansion of [16, Theorem 3.2], we achieve

$$\begin{aligned}\mathbf{M}(\tilde{T}_k) - \mathbf{M}(\bar{T}_k) \le \frac{1}{2}\int_{\mathcal{B}_\rho} &\Big(|D\tilde{N}_k|^2 - |D\bar{N}_k|^2\Big) \\ &+ C\|H_k\|_{C^0}\int \Big(|\boldsymbol{\eta}\circ\bar{N}_k| + |\boldsymbol{\eta}\circ\tilde{N}_k|\Big) \\ &+ \|A_k\|^2_{C^0}\int\Big(|\bar{N}_k|^2 + |\tilde{N}_k|^2\Big) + o(\mathbf{h}_k^2) \\ \le -\frac{\delta}{2}\mathbf{h}_k^2 + o(\mathbf{h}_k^2)\,. \end{aligned} \tag{7.18}$$

Clearly, (7.18) and (7.17) contradict the minimizing property of $\bar{T}_k$ for k large enough and this concludes the proof.

7.2. **Persistence of singularities.** We discuss step (G) of § 2.7: we show that the assumptions (7.2) and (7.3) contradict Theorem 3.1.2, which asserts that the singular set of N_∞^b has $\mathcal{H}^{m-2+\alpha}$ measure zero.

Set

$$\Upsilon := \Big\{x \in \bar{B}_1 : N_\infty^b(x) = Q\,[\![0]\!]\Big\},$$

and note that, since $\boldsymbol{\eta}\circ N_\infty^b \equiv 0$ and $\|N_\infty^b\|_{L^2(B_{\frac{3}{2}})} = 1$, from Theorem 3.1.2 it follows that $\mathcal{H}_\infty^{m-2+\alpha}(\Upsilon) = 0$.

The main line of the contradiction argument can be summarized in three steps.

(1) By (7.2) and (7.3) there exists a set $\Lambda_k \subset \mathrm{Dir}_Q(\bar{T}_k)$ such that

$$\mathrm{dist}(\Lambda_k, \Upsilon) > c_1 > 0 \quad \text{and} \quad \mathcal{H}_\infty^{m-2+\alpha}(\Lambda_k) > c_2 > 0,$$

for suitable constants $c_1, c_2 > 0$.

The key aspect of the set Λ_k is the following: by the Hölder regularity of Dir-minimizing functions in Theorem 3.1.2, the normal approximation $\bar{N}_k$ must be big in modulus around any point in Λ_k.

(2) Moreover, it follows from the Lipschitz approximation Theorem 3.3.2 (see Theorem 7.2.2 below that around any multiplicity Q point of the current the energy of the Lipschitz approximation is large enough with respect to the L^2 norm (cp. [17, Theorem 1.7]). This is what we call *persistence of Q-point* phenomenon, and is in fact the analytic core of this part of the proof.

We moreover stress that this part of the proof (even if it is not apparent from our exposition) also uses the *splitting-before-tilting* estimates.

(3) Putting together the previous two steps, we then conclude that there is a big part of the current where the energy of the Lipschitz approximation is large enough: matching the constant in the previous estimates, one realizes that this cannot happen on a set of positive $\mathcal{H}^{m-2+\alpha}$ measure.

As usual, the actual proof is much more involved of the heuristic scheme above. In the following we try to give some more explanations, referring to [17, 14, 15] for the detailed proof.

Step (1). We cover Υ by balls $\{\mathbf{B}_{\sigma_i}(x_i)\}$ in such a way that

$$\sum_i \omega_{m-2+\alpha}(4\sigma_i)^{m-2+\alpha} \leq \frac{\eta'}{2},$$

where $\eta' > 0$ is the constant in (7.8). By the compactness of Υ, such a covering can be chosen finite. Let $\sigma > 0$ be a radius whose specific choice will be given only at the very end, and such that $0 < 40\,\sigma \leq \min \sigma_i$. Denote by Λ_k the set of Q points of $\bar{T}_k$ far away from the singular set Υ:

$$\Lambda_k := \big\{p \in D_Q(\bar{T}_k) \cap \mathbf{B}_1 : \mathrm{dist}(p, \Upsilon) > 4 \min \sigma_i\big\}.$$

Clearly, $\mathcal{H}^{m-2+\alpha}_\infty(\Lambda_k) \geq \frac{\eta'}{2}$. Let $\mathbf{V}$ denote the neighborhood of Υ of size $2 \min \sigma_i$. By the Hölder continuity of Dir-minimizing functions in Theorem 3.1.2 (ii), there is a positive constant $\vartheta > 0$ such that $|N^b_\infty(x)|^2 \geq 2\vartheta$ for every $x \notin \mathbf{V}$. It then follows that

$$2\,\vartheta \leq \fint_{B_{2\sigma}(x)} |N^b_\infty|^2 \qquad \forall\, x \in B_{\frac{5}{4}} \text{ with } \mathrm{dist}(x, \Upsilon) \geq 3 \min \sigma_i,$$

and therefore, for sufficiently large k's,

$$\vartheta\, \mathbf{h}_k^2 \leq \fint_{\mathcal{B}_{2\sigma}(x)} \mathcal{G}(\bar{N}_k, Q\, [\![\boldsymbol{\eta} \circ \bar{N}_k]\!])^2, \tag{7.19}$$

for all $x \in \Gamma_k := \mathbf{p}_{\bar{\mathcal{M}}_k}(\Lambda_k)$. This is the claimed lower bound on the modulus of $\bar{N}_k$.

Step (2). This is the most important step of the proof. We start introducing the following notation. For every $p \in \Lambda_k$, consider $\bar{z}_k(p) = \mathbf{p}_{\pi_0}(p)$ and $\bar{x}_k(p) := \bar{\mathbf{\Phi}}_k \in \bar{\mathcal{M}}_k$, where $\bar{\mathbf{\Phi}}_k$ is the induced parametrization.

The key claim is the following: there exists a geometric constant $c_0 > 0$ (in particular, independent of σ) such that, when k is large enough, for each $p \in \Lambda_k$ there is a radius $\varrho_p \leq 2\sigma$ with the following properties:

$$\frac{c_0\, \vartheta}{\sigma^\alpha} \mathbf{h}_k^2 \leq \frac{1}{\varrho_p^{m-2+\alpha}} \int_{\mathcal{B}_{\varrho_p}(\bar{x}_k(p))} |D\bar{N}_k|^2, \tag{7.20}$$

$$\mathcal{B}_{\varrho_p}(\bar{x}_k(p)) \subset \mathbf{B}_{4\varrho_p}(p)\,. \tag{7.21}$$

We show here the main heuristics leading to (7.20) (and we warn the reader that these are not the complete arguments), referring to [15] for (7.21). The key estimate in this regard is the following: there exists a constant $\bar{s} < 1$ such that

$$\fint_{\mathcal{B}_{\bar{s}\ell(L_k)}(x_k)} \mathcal{G}(N_{j(k)}, Q\,[\![\boldsymbol{\eta} \circ N_{j(k)}]\!])^2 \leq \frac{\vartheta}{4\omega_m \ell(L_k)^{m-2}} \int_{\mathcal{B}_{\ell(L_k)}(x_k)} |DN_{j(k)}|^2\,,$$

that is, rescaling to $\bar{\mathcal{M}}_k$, there exists $t(p) \leq \bar{\ell}_k$ such that

$$\fint_{\mathcal{B}_{\bar{s}t(p)}(\bar{x}_k)(p)} \mathcal{G}(\bar{N}_k, Q\,[\![\boldsymbol{\eta} \circ \bar{N}_k]\!])^2 \leq \frac{\vartheta}{4\omega_m t(p)^{m-2}} \int_{\mathcal{B}_{t(p)}(\bar{x}_k(p))} |D\bar{N}_k|^2\,. \tag{7.22}$$

We show that we can choose $\varrho_p \in]\bar{s}\,t(p), 2\sigma[$ such that (7.20) follows from (7.22). To this aim we can distinguish two cases. Either

$$\frac{1}{\omega_m t(p)^{m-2}} \int_{\mathcal{B}_{t(p)}(\bar{x}_k(p))} |DN_k|^2 \geq \mathbf{h}_k^2\,, \tag{7.23}$$

and (7.20) follows with $\varrho_p = t(p)$. Or (7.23) does not hold, and we argue as follows. We use first (7.22) to get

$$\fint_{\mathcal{B}_{\bar{s}t(p)}(\bar{x}_k(p))} \mathcal{G}(\bar{N}_k, Q\,[\![\boldsymbol{\eta} \circ \bar{N}_k]\!])^2 \leq \frac{\vartheta}{4}\mathbf{h}_k^2\,. \tag{7.24}$$

Then, we show by contradiction that there exists a radius $\varrho_y \in [\bar{s}t(p), 2\sigma]$ such that (7.20) holds. Indeed, if this were not the case, setting for simplicity $f := \mathcal{G}(\bar{N}_k, Q\,[\![\boldsymbol{\eta} \circ \bar{N}_k]\!])$ and letting j be the smallest integer such that $2^{-j}\sigma \leq \bar{s}t(p)$, we can estimate as follows

$$\begin{aligned}
\fint_{\mathcal{B}_{2\sigma}(\bar{x}_k(p))} f^2 &\leq 2\fint_{\mathcal{B}_{\bar{s}t(p)}(\bar{x}_k(p))} f^2 + \sum_{i=0}^{j}\left(\fint_{\mathcal{B}_{2^{1-i}\sigma}(\bar{x}_k(p))} f^2 - \fint_{\mathcal{B}_{2^{-i}\sigma}(\bar{x}_k(p))} f^2\right)\\
&\overset{(7.24)}{\leq} \frac{\vartheta}{2}\mathbf{h}_k^2 + C\sum_{i=1}^{j}\frac{1}{(2^{-j}\sigma)^{m-2}}\int_{\mathcal{B}_{2^{1-i}\sigma}(\bar{x}_k(p))} |D\bar{N}_k|^2\\
&\leq \frac{\vartheta}{2}\mathbf{h}_k^2 + Cc_0\frac{\vartheta}{\sigma^\alpha}\mathbf{h}_k^2\sum_{i=1}^{j}(2^{-j}\sigma)^\alpha \leq \mathbf{h}_k^2\left(\frac{\vartheta}{2} + C(\alpha)c_0\vartheta\right).
\end{aligned}$$

In the second line we have used the simple Morrey inequality

$$\begin{aligned}
\left|\fint_{\mathcal{B}_{2t}(\bar{x}_k(p))} f^2 - \fint_{\mathcal{B}_{t}(\bar{x}_k(p))} f^2\right| &\leq \frac{C}{t^{m-2}}\int_{\mathcal{B}_{2t}(\bar{x}_k(p))} |Df|^2\\
&\leq \frac{C}{t^{m-2}}\int_{\mathcal{B}_{2t}(\bar{x}_k(p))} |D\bar{N}_k|^2\,.
\end{aligned}$$

The constant C depends only upon the regularity of the underlying manifold $\bar{\mathcal{M}}_k$, and, hence, can assumed independent of k.

Since $C(\alpha)$ depends only on α, m and Q, for c_0 chosen sufficiently small the latter inequality would contradict (7.19).

Step (3). We collect the estimates (7.20) and (7.21) to infer the desired contradiction. We cover Λ_k with balls $\mathbf{B}^i := \mathbf{B}_{20\varrho_{p_i}}(p_i)$ such that $\mathbf{B}_{4\varrho_{p_i}}(p_i)$ are disjoint, and deduce

$$\frac{\eta'}{2} \leq C(m) \sum_i \varrho_{p_i}^{m-2+\alpha} \overset{(7.20)}{\leq} \frac{C(m)}{c_0} \frac{\sigma^\alpha}{\vartheta \mathbf{h}_k^2} \sum_i \int_{\mathcal{B}_{\varrho_{p_i}}(\bar{x}_k(p_i))} |D\bar{N}_k|^2$$
$$\leq \frac{C(m)}{c_0} \frac{\sigma^\alpha}{\vartheta \mathbf{h}_k^2} \int_{\mathcal{B}_{\frac{3}{2}}} |D\bar{N}_k|^2 \overset{(7.10)}{\leq} C\frac{\sigma^\alpha}{\vartheta},$$

where $C(m) > 0$ is a dimensional constant. We have used that the balls $\mathcal{B}_{\varrho_{p_i}}(\mathbf{p}_{\bar{\mathcal{M}}_k}(p_i))$ are pairwise disjoint by (7.21). Now note that ϑ and c_0 are independent of σ, and therefore we can finally choose σ small enough to lead to a contradiction.

7.2.1. *Persistence of Q-points.* Here we explain a simple instance of estimate (7.22), reporting the following theorem from [17].

Theorem 7.2.2 (Persistence of Q-points)**.** *For every $\hat{\delta} > 0$, there is $\bar{s} \in]0, \frac{1}{2}[$ such that, for every $s < \bar{s}$, there exists $\hat{\varepsilon}(s, \hat{\delta}) > 0$ with the following property. If T is as in Theorem 3.3.2, $E := \mathbf{E}(T, \bar{C}_{4\,r}(x)) < \hat{\varepsilon}$ and $\Theta(T,(p,q)) = Q$ at some $(p,q) \in \bar{C}_{r/2}(x)$, then the approximation f of Theorem 3.3.2 satisfies*

$$\int_{B_{sr}(p)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta} \circ f]\!])^2 \leq \hat{\delta} s^m r^{2+m} E\,. \tag{7.25}$$

This theorem states that, in the presence of multiplicity Q points of the current, the Lipschitz (and therefore also the normal) approximations must have a relatively small L^2 norm, compared to the excess; or, as explained above, if in the normal approximation the excess is linked to the Dirichlet energy (for example this is the case of (EX)-cubes in the Whitney decomposition), the energy needs to be relatively large with respect to the L^2 norm, thus vaguely explaining the link to (7.22).

Proof. By scaling and translating we assume $x = 0$ and $r = 1$; the choice of $\bar{s}$ will be specified at the very end, but for the moment we impose $\bar{s} < \frac{1}{4}$. Assume by contradiction that, for arbitrarily small $\hat{\varepsilon} > 0$, there are currents T and points $(p,q) \in \bar{C}_{1/2}$ satisfying: $E := \mathbf{E}(T, \bar{C}_4) < \hat{\varepsilon}$, $\Theta(T,(p,q)) = Q$ and, for f as in Theorem 3.3.2,

$$\int_{B_s(p)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta} \circ f]\!])^2 > \hat{\delta} s^m E\,. \tag{7.26}$$

Set $\bar{\delta} = \frac{1}{4}$ and fix $\bar{\eta} > 0$ (whose choice will be specified later). For a suitably small $\hat{\varepsilon}$ we can apply Theorem 3.3.3, obtaining a Dir-minimizing approximation w. If $\bar{\eta}$ and $\hat{\varepsilon}$ are suitably small, we have

$$\int_{B_s(p)} \mathcal{G}(w, Q\,[\![\boldsymbol{\eta} \circ w]\!])^2 \geq \tfrac{3\hat{\delta}}{4} s^m E\,,$$

and $\sup\{\mathrm{Dir}(f), \mathrm{Dir}(w)\} \leq CE$. Then there exists $\bar p \in B_s(p)$ with

$$\mathcal{G}(w(\bar p), Q\,[\![\boldsymbol{\eta}\circ w(\bar p)]\!])^2 \geq \frac{3\hat\delta}{4\omega_m}\,E,$$

and, by the Hölder continuity in Theorem 3.1.2 (ii), we conclude

$$\begin{aligned} g(x) &:= \mathcal{G}(w(x), Q\,[\![\boldsymbol{\eta}\circ w(x)]\!]) \\ &\geq \left(\tfrac{3\hat\delta}{4\omega_m}E\right)^{\frac12} - 2\,(CE)^{\frac12}\bar C \bar s^{\kappa} \geq \left(\tfrac{\hat\delta}{2}E\right)^{\frac12}\,, \end{aligned} \tag{7.27}$$

where we assume that $\bar s$ is chosen small enough in order to satisfy the last inequality. Setting $h(x) := \mathcal{G}(f(x), Q\,[\![\boldsymbol{\eta}\circ f(x)]\!])$, we recall that we have

$$\int_{B_s(p)} |h-g|^2 \leq C\,\bar\eta E\,.$$

Consider therefore the set $A := \left\{h > \left(\tfrac{\hat\delta}{4}E\right)^{\frac12}\right\}$. If $\bar\eta$ is sufficiently small, we can assume that

$$|B_s(p)\setminus A| < \frac18 |B_s|.$$

Further, define $\bar A := A\cap K$, where K is the set of Theorem 3.3.2. Assuming $\hat\varepsilon$ is sufficiently small we ensure $|B_s(p)\setminus \bar A| < \frac14|B_s|$. Let N be the smallest integer such that $N\frac{\hat\delta E}{64Qs} \geq \frac{s}{2}$. Set

$$\sigma_i := s - i\frac{\hat\delta E}{64Qs} \quad \text{for } i\in\{0,1\ldots,N\},$$

and consider, for $i\leq N-1$, the annuli $\mathcal{C}_i := B_{\sigma_i}(p)\setminus B_{\sigma_{i+1}}(p)$. If $\hat\varepsilon$ is sufficiently small, we can assume that $N\geq 2$ and $\sigma_N \geq \frac{s}{4}$. For at least one of these annuli we must have $|\bar A\cap \mathcal{C}_i| \geq \frac12|\mathcal{C}_i|$. We then let $\sigma := \sigma_i$ be the corresponding outer radius and we denote by $\mathcal{C}$ the corresponding annulus.

Consider now a point $x\in\mathcal{C}\cap\bar A$ and let T_x be the slice $\langle T, \mathbf{p}, x\rangle$. Since $\bar A\subset K$, for a.e. $x\in\bar A$ we have $T_x = \sum_{i=1}^Q [\![(x, f_i(x))]\!]$. Moreover, there exist i and j such that $|f_i(x)-f_j(x)|^2 \geq \frac1Q\mathcal{G}(f(x), [\![\boldsymbol{\eta}\circ f(x)]\!])^2 \geq \frac{\hat\delta}{4Q}E$ (recall that $x\in\bar A\subset A$). When $x\in\mathcal{C}$ and the points (x,y) and (x,z) belong both to $\mathbf{B}_\sigma((p,q))$, we must have

$$|y-z|^2 \leq 4\left(\sigma^2 - \left(\sigma - \tfrac{\hat\delta E}{64Qs}\right)^2\right) \leq \tfrac{\sigma\hat\delta E}{8Qs} \leq \tfrac{\hat\delta E}{8Q}\,.$$

Thus, for $x\in\bar A\cap\mathcal{C}$ at least one of the points $(x, f_i(x))$ is not contained in $\mathbf{B}_\sigma((p,q))$. We conclude therefore

$$\begin{aligned} \|T\|(\bar C_\sigma(p)\setminus \mathbf{B}_\sigma((p,q))) &\geq |\mathcal{C}\cap\bar A| \geq \frac12|\mathcal{C}| = \frac{\omega_m}{2}\left(\sigma^m - \left(\sigma - \tfrac{\hat\delta E}{64Qs}\right)^m\right) \\ &\geq \frac{\omega_m}{2}\sigma^m\left(1 - \left(1 - \tfrac{\hat\delta E}{64Qs\sigma}\right)^m\right)\,. \end{aligned} \tag{7.28}$$

Recall that, for τ sufficiently small, $(1-\tau)^m \leq 1 - \frac{m\tau}{2}$. Since $\sigma \geq \frac{s}{4}$, if $\hat{\varepsilon}$ is chosen sufficiently small we can therefore conclude

$$\|T\|(\bar{C}_\sigma(p) \setminus \mathbf{B}_\sigma(p)) \geq \frac{\omega_m \sigma^m \hat{\delta} E}{256 Q s \sigma} \geq \frac{\omega_m}{1024 Q} \hat{\delta} E \sigma^{m-2} = c_0 \hat{\delta} E \sigma^{m-2}\,. \quad (7.29)$$

Next, by Theorem 3.3.2 and Theorem 3.3.3,

$$\|T\|(\bar{C}_\sigma(p)) \leq Q\omega_m \sigma^m + C E^{1+\gamma_1} + \bar{\eta} E + \int_{B_\sigma(p)} \frac{|Dw|^2}{2}\,. \quad (7.30)$$

Moreover, as shown in [13, Proposition 3.10], we have

$$\int_{B_\sigma(p)} |Dw|^2 \leq C \mathrm{Dir}(w) \sigma^{m-2+2\kappa}, \quad (7.31)$$

(for some constants κ and C depending only on m, n and Q; in fact the exponent κ is the one of Theorem 3.1.2 (ii)). Combining (7.29), (7.30) and (7.31), we conclude

$$\|T\|(\mathbf{B}_\sigma((p,q))) \leq Q\omega_m \sigma^m + \bar{\eta}\, E + C E^{1+\gamma_1} + C E \sigma^{m-2+2\kappa} - c_0 \sigma^{m-2} \hat{\delta} E\,. \quad (7.32)$$

Next, by the monotonicity formula, $\rho \mapsto \rho^{-m} \|T\|(\mathbf{B}_\rho((p,q)))$ is a monotone function. Using $\Theta(T,(p,q)) = Q$, we conclude

$$\|T\|(\mathbf{B}_\sigma((p,q))) \geq Q\omega_m \sigma^m. \quad (7.33)$$

Combining (7.32) and (7.33) we conclude

$$C\sigma^2 + (\bar{\eta} + C E_1^{\gamma}) \sigma^{2-m} + C\sigma^{2\kappa} \geq c_0 \hat{\delta}\,. \quad (7.34)$$

Recalling that $\sigma \leq s < \bar{s}$, we can, finally, specify $\bar{s}$: it is chosen so that $C\bar{s}^2 + C\bar{s}^{2\kappa}$ is smaller than $\frac{c_0}{2}\hat{\delta}$. Combined with (7.27) this choice of $\bar{s}$ depends, therefore, only upon $\hat{\delta}$. (7.34) becomes then

$$(\bar{\eta} + C E^{\gamma_1}) \sigma^{2-m} \geq \tfrac{c_0}{2} \hat{\delta}\,. \quad (7.35)$$

Next, recall that $\sigma \geq \frac{s}{4}$. We then choose $\hat{\varepsilon}$ and $\bar{\eta}$ so that $(\bar{\eta} + C\hat{\varepsilon}^{\gamma_1})(\frac{s}{4})^{2-m} \leq \frac{c_0}{4}\hat{\delta}$. This choice is incompatible with (7.35), thereby reaching a contradiction: for this choice of the parameter $\hat{\varepsilon}$ (which in fact depends only upon $\hat{\delta}$ and s) the conclusion of the theorem, i.e. (7.25), must then be valid. □

8. Open questions

We close this survey recalling some open problems concerning the regularity of area minimizing integer rectifiable currents. Some of them have been only slightly touched and would actually explain some of the complications that we met along the proof of the partial regularity result.

For more open problems and comments, we suggest the reading of [1, 11].

(A). One of the main, perhaps the most well-known, open problems is the uniqueness of the tangent cones to an area minimizing current, i.e. the uniqueness of the limit $(\iota_{x,r})_\sharp T$ as $r \to 0$ for every $x \in \mathrm{spt}\,(T)$. The uniqueness is known for two dimensional currents (cp. [35]), and there are only partial results in the general case (see [4, 30]).

We have run into this issue in dealing with the step (C) of § 2.7, because it is one of the possible reasons why a center manifold may be sufficient in our proof.

(B). A related question is that of the uniqueness of the inhomogeneous blowup for Dir-minimizing Q-valued functions. Also in this case the uniqueness is known for two dimensional domains (cp. [13], following ideas of [7]).

Even if it does not play a role in the contradiction argument for the partial regularity, a positive answer to this question could indeed contribute to the solution of next two other major open problems.

(C). It is unknown whether the singular set of an area minimizing current has always locally finite $\mathcal{H}^{m-2}$ measure. This is the case for two dimensional currents (as proven by Chang [7]); note that in this result the uniqueness of the blowup Dir-minimizing map plays a fundamental role.

(D). It is unknown whether the singular set of an area minimizing current has some geometric structure, e.g. if it is rectifiable (i.e., roughly speaking, if it is contained in lower dimensional $(m-2)$-dimensional submanifolds). Once again it is known the positive answer for two dimensional currents, where the singularities are known to be locally isolated, and the uniqueness of the tangent map is one of the fundamental steps in the proof.

(E). We mention also the problem of finding more example of area minimizing currents, other than those coming from complex varieties or similar calibrations. Indeed, our understanding of the possible pathological behaviors of such currents is pretty much limited by the few examples we have at disposal. In particular, it would be extremely interesting to understand if there could be minimizing currents with weird singular set (e.g., of Cantor type).

(F). Finally, we mention the problem of boundary regularity for higher codimension area minimizing currents, which to our knowledge is mostly open.

References

[1] Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute. In J. E. Brothers, editor, *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 441–464. Amer. Math. Soc., Providence, RI, 1986.

[2] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[3] William K. Allard. On the first variation of a varifold: boundary behavior. *Ann. of Math. (2)*, 101:418–446, 1975.

[4] William K. Allard and Frederick J. Almgren, Jr. On the radial behavior of minimal surfaces and the uniqueness of their tangent cones. *Ann. of Math. (2)*, 113(2):215–265, 1981.
[5] Frederick J. Almgren, Jr. *Almgren's big regularity paper*, volume 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
[6] Luigi Ambrosio. Metric space valued functions of bounded variation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(3):439–478, 1990.
[7] Sheldon Xu-Dong Chang. Two-dimensional area minimizing integral currents are classical minimal surfaces. *J. Amer. Math. Soc.*, 1(4):699–778, 1988.
[8] Ennio De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.
[9] Ennio De Giorgi. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. *Ricerche Mat.*, 4:95–113, 1955.
[10] Ennio De Giorgi. *Frontiere orientate di misura minima*. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61. Editrice Tecnico Scientifica, Pisa, 1961.
[11] Camillo De Lellis. Almgren's Q-valued functions revisited. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1910–1933. Hindustan Book Agency, New Delhi, 2010.
[12] Camillo De Lellis and Emanuele Spadaro. Center manifold: a case study. *Discrete Contin. Dyn. Syst.*, 31(4):1249–1272, 2011.
[13] Camillo De Lellis and Emanuele Spadaro. Q-valued functions revisited. *Mem. Amer. Math. Soc.*, 211(991):vi+79, 2011.
[14] Camillo De Lellis and Emanuele Spadaro. Regularity of area-minimizing currents II: center manifold. *Preprint*, 2013.
[15] Camillo De Lellis and Emanuele Spadaro. Regularity of area-minimizing currents III: blow-up. *Preprint*, 2013.
[16] Camillo De Lellis and Emanuele Spadaro. Multiple valued functions and integral currents. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, to appear (2014).
[17] Camillo De Lellis and Emanuele Spadaro. Regularity of area-minimizing currents I: gradient L^p estimates. *GAFA*, to appear (2014).
[18] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
[19] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.
[20] Wendell H. Fleming. On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)*, 11:69–90, 1962.
[21] Matteo Focardi, Andrea Marchese, and Emanuele Spadaro. Improved estimate of the singular set of Dir-minimizing Q-valued functions. *Preprint*, 2014.
[22] Mikhail Gromov and Richard Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.*, (76):165–246, 1992.
[23] Robert Hardt and Leon Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. of Math. (2)*, 110(3):439–486, 1979.
[24] Jonas Hirsch. Boundary regularity of Dirichlet minimizing Q-valued fucntions. *Preprint*, 2014.
[25] Jürgen Jost. Generalized Dirichlet forms and harmonic maps. *Calc. Var. Partial Differential Equations*, 5(1):1–19, 1997.
[26] Nicholas J. Korevaar and Richard M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.

[27] Philippe Logaritsch and Emanuele Spadaro. A representation formula for the p-energy of metric space-valued Sobolev maps. *Commun. Contemp. Math.*, 14(6):1250043, 10, 2012.

[28] E. R. Reifenberg. On the analyticity of minimal surfaces. *Ann. of Math. (2)*, 80:15–21, 1964.

[29] Tristan Rivière. A lower-epiperimetric inequality for area-minimizing surfaces. *Comm. Pure Appl. Math.*, 57(12):1673–1685, 2004.

[30] Leon Simon. Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. *Ann. of Math. (2)*, 118(3):525–571, 1983.

[31] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[32] Leon Simon. Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 246–305. Int. Press, Cambridge, MA, 1995.

[33] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.

[34] Emanuele Spadaro. Complex varieties and higher integrability of Dir-minimizing Q-valued functions. *Manuscripta Math.*, 132(3-4):415–429, 2010.

[35] Brian White. Tangent cones to two-dimensional area-minimizing integral currents are unique. *Duke Math. J.*, 50(1):143–160, 1983.

E-mail address: spadaro@mis.mpg.de